Midterm Exam MATH 205
Differential and Integral Calculus II

Winter 2018 **March 3, 2018** **Time allowed: 1h 15min**

SOLUTIONS

Problem 1.

(b) Use the Fundamental Theorem of Calculus to calculate the derivative of

$$F(x) = \int_{-x^2}^{0} (t-1)\cos^4(t+1)dt$$

and determine whether F is increasing or decreasing at $x = 1/2$?

Solution.

By the Fundamental Theorem of Calculus

$$F'(x) = \frac{d}{dx}\left(\int_{-x^2}^{0} (t-1)\cos^4(t+1)dt\right) = -\frac{d}{dx}\left(\int_{0}^{-x^2} (t-1)\cos^4(t+1)dt\right) =$$

$$= -(-x^2-1)\cos^4(-x^2+1)\frac{d}{dx}(-x^2) = -2x(x^2+1)\cos^4(1-x^2).$$

Since $F'(1/2) = -2 \cdot \frac{1}{2} \cdot \frac{5}{4}\cos^4\left(\frac{3}{4}\right) < 0$, then F decreases at $x = 1/2$.

Problem 2.

Find $G(x)$ if $G'(x) = \sec^3(x)\tan^3(x)$ and $G(0) = 1$.

Solution.

$$G(x) = \int \sec^3(x)\tan^3(x)dx = \int \sec^2(x)\tan^2(x)(\sec(x)\tan(x))dx =$$

$$= \int \sec^2(x)(\sec^2(x)-1)(\sec(x)\tan(x))dx = \int u^2(u^2-1)du,$$

where $u = \sec(x)$, $du = \sec(x)\tan(x)dx$.
Therefore,

$$G(x) = \int u^2(u^2-1)du = \int (u^4-u^2)du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\sec^5(x) - \frac{1}{3}\sec^3(x) + C.$$

From $G(0) = \frac{1}{5}\sec^5(0) - \frac{1}{3}\sec^3(0) + C = \frac{1}{5} - \frac{1}{3} + C = C - \frac{2}{15} = 1$ we get $C = 1 + \frac{2}{15} = \frac{17}{15}$ and

$$G(x) = \frac{1}{5}\sec^5(x) - \frac{1}{3}\sec^3(x) + \frac{17}{15}.$$

Problem 3.

Calculate the following indefinite integrals

(a) $\int \ln\sqrt{x}dx$

Solution.

By the integration by parts

$$\int \ln\sqrt{x}dx = \frac{1}{2}\int \ln x dx = \frac{1}{2}\left(x\ln x - \int x d(\ln x)\right) = \frac{1}{2}\left(x\ln x - \int x\frac{1}{x}dx\right) =$$

$$= \frac{1}{2}\left(x\ln x - \int dx\right) = \frac{1}{2}(x\ln x - x) + C$$

(b) $\int \frac{x+1}{x^3+x}dx$

Solution.

We write the integrand as a sum of partial fractions

$$\frac{x+1}{x^3+x} = \frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

with unknown coefficients A, B and C which we find from the following identity

$$A(x^2+1) + (Bx+C)x \equiv x+1$$

From

$$(A+B)x^2 + Cx + A \equiv x+1$$

we get system of three linear algebraic equations for A, B and C

$$A+B=0, \quad C=1, \quad A=1$$

Therefore, $A=1$, $B=-1$ and $C=1$ and

$$\int \frac{x+1}{x^3+x}dx = \int\left(\frac{1}{x} - \frac{x-1}{x^2+1}\right)dx = \int\left(\frac{1}{x} - \frac{1}{2}\frac{2x}{x^2+1} + \frac{1}{x^2+1}\right)dx =$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \arctan x + C$$

Problem 4.

Find the exact value of the following definite integrals (do not approximate)

(a) $\int_0^a x\sqrt{a-x}dx$, where parameter $a > 0$

Solution.

By substitution $u = a - x$, $du = -dx$, $u(0) = a$, $u(a) = 0$ and $x = a - u$ we get

$$\int_0^a x\sqrt{a-x}dx = -\int_a^0 (a-u)\sqrt{u}du = \int_0^a (au^{1/2} - u^{3/2})du =$$

$$= \left(\frac{2}{3}au^{3/2} - \frac{2}{5}u^{5/2}\right)\Bigg|_0^a = \frac{2}{3}a^{5/2} - \frac{2}{5}a^{5/2} = \frac{4}{15}a^{5/2}$$

(b) $\int_0^{\pi/2}(2 - \sin(x))^2 dx$

Solution.

$$\int_0^{\pi/2}(2-\sin(x))^2 dx = \int_0^{\pi/2}(4-4\sin(x)+\sin^2(x))dx = \int_0^{\pi/2}\left(4 - 4\sin(x) + \frac{1-\cos(2x)}{2}\right)dx =$$

$$= \int_0^{\pi/2}\left(\frac{9}{2} - 4\sin(x) - \frac{1}{2}\cos(2x)\right)dx = \left(\frac{9}{2}x + 4\cos(x) - \frac{1}{4}\sin(2x)\right)\Bigg|_0^{\pi/2} = \frac{9}{4}\pi - 4$$

Problem 5.

Sketch the region bounded by the graph $f(x) = 4x - x^2$ and the x-axis, and find the volume of a solid obtained by revolution of this region about the axis $y = -2$.

Solution.

Points of intersection the graph with the x-axis $f(x) = 4x - x^2 = x(4-x) = 0$, i.e. $x = 0$ and $x = 4$. Distance from the parabola to the axis of rotation is $R(x) = f(x) - (-2) = 4x - x^2 + 2$, distance from the x-axis to the axis of rotation is $r(x) = 0 - (-2) = 2$. Therefore,

$$V = \int_0^4 \pi(R^2(x) - r^2(x))dx = \pi\int_0^4((2 + 4x - x^2)^2 - 2^2)dx =$$

$$= \pi\int_0^4(4+8x-2x^2+8x+16x^2-4x^3-2x^2-4x^3+x^4-4)dx = \pi\int_0^4(16x+12x^2-8x^3+x^4)dx =$$

$$= \pi(8x^2 + 4x^3 - 2x^4 + \frac{1}{5}x^5)|_0^4 = \frac{384\pi}{5}$$

Problem 6.

Find the average value of the function $f(x) = (x-3)^2$ on the interval $[2, 5]$. Sketch the graph of f, and draw the rectangle whose base is the interval $[2, 5]$ and whose height is the average value of f on that interval

Solution.

$$f_{AV} = \frac{1}{5-2}\int_2^5 f(x)dx = \frac{1}{3}\int_2^5 (x-3)^2 dx = \frac{1}{9}(x-3)^3|_2^5 = \frac{1}{9}((5-3)^3 - (2-3)^3) = 1$$

Bonus.

Evaluate the limit by recognizing it as a Riemann sum for a function $f(x)$ on the interval $[0, 1]$ and then using integration.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^4}{n^5} + \frac{i}{n^2}\right)$$

Solution.

$\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and partition $x_i = 0 + i\Delta x = \frac{i}{n}$, where $i = \overline{0, n}$.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^4}{n^5} + \frac{i}{n^2}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^4}{n^4} + \frac{i}{n}\right)\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^4}{n^4} + \frac{i}{n}\right)\Delta x =$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(x_i^4 + x_i\right)\Delta = \int_0^1 (x^4 + x)dx = \left(\frac{1}{5}x^5 + \frac{1}{2}x^2\right)\Bigg|_0^1 = \frac{1}{5} + \frac{1}{2} = \frac{7}{10}$$