Midterm Exam MATH 205
Differential and Integral Calculus II

Winter 2020 **March 8, 2020** **Time allowed: 1h 15min**

SOLUTIONS

Problem 1.

(a) Derive the sigma notation formula for the right Riemann sum R_n of the function $f(x) = x^2 + 2x$ on the interval $[-2, 0]$ using n subintervals of equal length. Then calculate the integral $\int_{-2}^{0} f(x)dx$ as the limit of R_n at $n \to \infty$.

Solution.

By partition $\Delta x = \frac{0-(-2)}{n} = \frac{2}{n}$, $x_i = -2 + \Delta x i = -2 + \frac{2}{n}i$. Then

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}((x_i)^2 + 2x_i)\Delta x = \sum_{i=1}^{n}\left((-2+\frac{2}{n}i)^2 + 2(-2+\frac{2}{n}i)\right)\frac{2}{n} =$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left(4 - 4\frac{2}{n}i + \frac{4}{n^2}i^2 - 4 + \frac{4}{n}i\right) = \frac{2}{n}\sum_{i=1}^{n}\left(\frac{4}{n^2}i^2 - \frac{4}{n}i\right) = \frac{8}{n^3}\sum_{i=1}^{n} i^2 - \frac{8}{n^2}\sum_{i=1}^{n} i =$$

$$= \frac{8}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{8}{n^2}\frac{n(n+1)}{2} = \frac{4}{3}\frac{(n+1)(2n+1)}{n^2} - 4\frac{n+1}{n}.$$

Therefore,

$$\int_{-2}^{0} f(x)dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(\frac{4}{3}\frac{(n+1)(2n+1)}{n^2} - 4\frac{n+1}{n}\right) =$$

$$= \lim_{n\to\infty}\left(\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 4\left(1+\frac{1}{n}\right)\right) = \frac{8}{3} - 4 = -\frac{4}{3}.$$

Verification

$$\int_{-2}^{0} f(x)dx = \int_{-2}^{0}(x^2+2x)dx = \left(\frac{1}{3}x^3 + x^2\right)\Big|_{-2}^{0}. = -\left(\frac{1}{3}(-2)^3 + (-2)^2\right) = \frac{8}{3} - 4 = -\frac{4}{3}.$$

(b) Define $F(x) = \int_{x}^{x^2} \frac{dt}{\ln(t)}$ for $x \geq 2$. Calculate $F'(x)$ for $x \geq 2$. Is F increasing or decreasing at $x = 2020$?

Solution.

By the Fundamental Theorem of Calculus

$$F'(x) = \frac{d}{dx}\int_{x}^{x^2}\frac{dt}{\ln(t)} = \frac{1}{\ln(x^2)}\frac{d}{dx}(x^2) - \frac{1}{\ln(x)} = \frac{2x}{2\ln(x)} - \frac{1}{\ln(x)} = \frac{x-1}{\ln(x)}.$$

Since $F'(2020) = \frac{2020-1}{\ln(2020)} = \frac{2019}{\ln(2020)} > 0$, then F increases at $x = 2020$.

Problem 2.

If $f'(x) = \frac{x+\arctan(x)}{x^2+1}$ and $f(0) = 1$, what is the exact value of $f(1)$?

Solution.

$$f(x) = \int \frac{x+\arctan(x)}{x^2+1}dx = \int \frac{x}{x^2+1}dx + \int \frac{\arctan(x)}{x^2+1}dx = \frac{1}{2}\int \frac{du}{u} + \int v dv$$

where $u = x^2+1$, $du = 2xdx$ and $v = \arctan(x)$, $dv = \frac{dx}{x^2+1}$.

Therefore,

$$f(x) = \frac{1}{2}\ln(u) + \frac{1}{2}v^2 + c = \frac{1}{2}\ln(x^2+1) + \frac{1}{2}(\arctan(x))^2 + c.$$

From $f(0) = \frac{1}{2}\ln(0^2+1) + \frac{1}{2}(\arctan(0))^2 + c = 0 + c = 1$,

$$f(x) = \frac{1}{2}\ln(x^2+1) + \frac{1}{2}(\arctan(x))^2 + 1.$$

Thus, $f(1) = \frac{1}{2}\ln 2 + \frac{1}{2}(\arctan 1)^2 + 1 = \frac{1}{2}\ln 2 + \frac{1}{2}\left(\frac{\pi}{4}\right)^2 + 1 = \frac{\ln 2}{2} + \frac{\pi^2}{32} + 1.$

Problem 3.

Calculate the following indefinite integrals

(a) $\int \frac{3x^2+x+2}{x(x^2+1)}dx$

Solution.

To use the method of partial fractions, we express the integrand as the sum of two fractions

$$\frac{3x^2+x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

with unknown coefficients A, B and C. From

$$\frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+x(Bx+C)}{x(x^2+1)} = \frac{(A+B)x^2+Cx+A}{x(x^2+1)} \equiv \frac{3x^2+x+2}{x(x^2+1)}$$

we get $(A+B)x^2+Cx+A \equiv 3x^2+x+2$ and thus, equations for unknowns A, B and C

$$A+B=3, \quad C=1 \quad A=2.$$

Therefore $C=1$, $A=2$ and $B=1$ and

$$\int \frac{3x^2+x+2}{x(x^2+1)}dx = \int \left(\frac{2}{x} + \frac{x+1}{x^2+1}\right)dx = \int \frac{2}{x}dx + \int \frac{x}{x^2+1}dx + \int \frac{1}{x^2+1}dx =$$

$$= 2\ln|x| + \frac{1}{2}\ln(x^2+1) + \arctan(x) + c.$$

(b) $\int (1+x^2)e^{-2x}dx$

Solution.

By integration by parts for $u = 1+x^2$, $du = 2xdx$ and $dv = e^{-2x}dx$, $v = -\frac{1}{2}e^{-2x}$, we get

$$\int (1+x^2)e^{-2x}dx = \int (1+x^2)d(-\frac{1}{2}e^{-2x}) = -\frac{1}{2}e^{-2x}(1+x^2) + \frac{1}{2}\int e^{-2x}d(1+x^2) =$$

$$= -\frac{1}{2}e^{-2x}(1+x^2) + \frac{1}{2}\int e^{-2x}2xdx = -\frac{1}{2}e^{-2x}(1+x^2) + \int e^{-2x}xdx.$$

Again by integration by parts for $u = x$, $du = dx$ and $dv = e^{-2x}$, $v = -\frac{1}{2}e^{-2x}$, we get

$$\int e^{-2x}xdx = \int xd(-\frac{1}{2}e^{-2x}) = -\frac{1}{2}e^{-2x}x + \frac{1}{2}\int e^{-2x}dx = -\frac{1}{2}e^{-2x}x - \frac{1}{4}e^{-2x} + c.$$

Therefore,

$$\int (1+x^2)e^{-2x}dx = -\frac{1}{2}e^{-2x}(1+x^2) - \frac{1}{2}e^{-2x}x - \frac{1}{4}e^{-2x} + c =$$

$$= \left(-\frac{1}{2}(x^2+1) - \frac{1}{2}x - \frac{1}{4}\right)e^{-2x} + c = \left(-\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}\right)e^{-2x} + c.$$

Problem 4.

Find the average value of the function $f(x) = \sin^3(x)\cos^7(x)$ on the interval $[0, \pi/2]$.

Solution.

$$f_{AV} = \frac{1}{\frac{\pi}{2} - 0}\int_0^{\pi/2} f(x)dx = \frac{2}{\pi}\int_0^{\pi/2} \sin^3(x)\cos^7(x)dx$$

By substitution $u = \cos(x)$, $du = -\sin(x)dx$, $u(0) = \cos 0 = 1$, $u(\frac{\pi}{2}) = \cos(\pi/2) = 0$ and from main trigonometric identity $\sin^2(x) + \cos^2(x) = 1$, we get

$$f_{AV} = \frac{2}{\pi}\int_0^{\pi/2} \sin^3(x)\cos^7(x)dx = \frac{2}{\pi}\int_0^{\pi/2} \sin^2(x)\cos^7(x)\sin(x)dx =$$

$$= \frac{2}{\pi}\int_0^{\pi/2} (1-\cos^2(x))\cos^7(x)\sin(x)dx = -\frac{2}{\pi}\int_1^0 (1-u^2)u^7du = \frac{2}{\pi}\int_0^1 (u^7 - u^9)du =$$

$$= \left(\frac{u^8}{8} - \frac{u^{10}}{10}\right)|_0^1 = \frac{1}{8} - \frac{1}{10} = \frac{1}{40}.$$

Problem 5.

Evaluate the following definite integrals

(a) $\int_0^3 \frac{3x+4}{\sqrt{x+1}}dx$

Solution.

By substitution $u = x + 1$, $du = dx$, $u(0) = 1$, $u(3) = 4$ and $x = u - 1$ we get

$$\int_0^3 \frac{3x+4}{\sqrt{x+1}}dx = \int_1^4 \frac{3(u-1)+4}{\sqrt{u}}du = \int_1^4 \frac{3u+1}{\sqrt{u}}du = \int_1^4 (3u^{1/2} + u^{-1/2})du =$$

$$= (2u^{3/2} + 2u^{1/2})|_1^4 = 2(4^{3/2} + 4^{1/2}) - 2(1^{3/2} + 1^{1/2}) = 20 - 4 = 16.$$

(b) $\int_0^{\pi/3} \tan^3(x)\sec(x)dx$

Solution.

By substitution $u = \sec(x)$, $du = \sec(x)\tan(x)dx$, $\sec 0 = 1$, $\sec(\pi/3) = 2$ and trigonometric identity $1 + \tan^2(x) = \sec^2(x)$ we get

$$\int_0^{\pi/3} \tan^3(x)\sec(x)dx = \int_0^{\pi/3} \tan^2(x)\sec(x)\tan(x)dx = \int_0^{\pi/3} (\sec^2(x) - 1)\sec(x)\tan(x)dx =$$

$$= \int_1^2 (u^2 - 1)du = (\frac{u^3}{3} - u)|_1^2 = \left(\frac{2^3}{3} - 2\right) - \left(\frac{1^3}{3} - 1\right) = \frac{4}{3}.$$

Problem 6.

Find the volume of a solid obtained by rotating the region enclosed by the curve $y = 2 - \sqrt{x}$ and the lines $y = 2$ and $x = 4$ about the x-axis.

Solution.

$$V = \int_0^4 \pi(R^2(x) - r^2(x))dx = \pi\int_0^4 (2^2 - (2 - \sqrt{x})^2)dx = \pi\int_0^4 (4 - (4 - 4\sqrt{x} + x))dx =$$

$$= \pi\int_0^4 (4 - 4 + 4\sqrt{x} - x)dx = \pi\int_0^4 (4\sqrt{x} - x)dx = \left(\frac{8}{3}x^{3/2} - \frac{1}{2}x^2\right)|_0^4 = \frac{8}{3}4^{3/2} - \frac{1}{2}4^2 = \frac{40}{3}.$$

Bonus.

Let $F(x) = \left(1 + \int_1^x e^{-t^2} dt\right)^2$. Calculate $F''(x)$ to determine whether the graph of F is concave upward or concave downward at $x = 1$.

Solution.

By the Chain Rule and the Fundamental Theorem of Calculus,

$$F'(x) = \frac{d}{dx}\left(\left(1 + \int_1^x e^{-t^2} dt\right)^2\right) = 2\left(1 + \int_1^x e^{-t^2} dt\right)\frac{d}{dx}\left(1 + \int_1^x e^{-t^2} dt\right) =$$

$$= 2\left(1 + \int_1^x e^{-t^2} dt\right)e^{-x^2}$$

and by the Product Rule

$$F''(x) = 2\frac{d}{dx}\left(\left(1 + \int_1^x e^{-t^2} dt\right)e^{-x^2}\right) = 2\frac{d}{dx}\left(1 + \int_1^x e^{-t^2} dt\right)e^{-x^2} + 2\left(1 + \int_1^x e^{-t^2} dt\right)\frac{d}{dx}(e^{-x^2}) =$$

$$= 2e^{-2x^2} - 4xe^{-x^2}\left(1 + \int_1^x e^{-t^2} dt\right).$$

Therefore,

$$F''(1) = 2e^{-2} - 4e^{-1}\left(1 + \int_1^1 e^{-t^2} dt\right) = 2e^{-2} - 4e^{-1} = 2e^{-2}(1 - 2e) < 0$$

and F is concave downward at $x = 1$.