LEC 2 – Even/Odd Functions and Rational Functions

Even and Odd Functions

Definition 2.1. (Even/Odd Functions) Let f be a fcn.

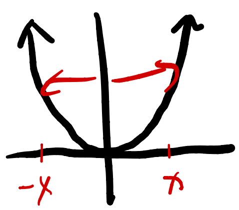
f is <u>even</u> if $f(-x) = f(x)$ for all x in the domain.

f is <u>odd</u> if $f(-x) = -f(x)$ for all x in the domain.

(Note: most fcns are neither and some fcns can be both.)

Example 2.2.

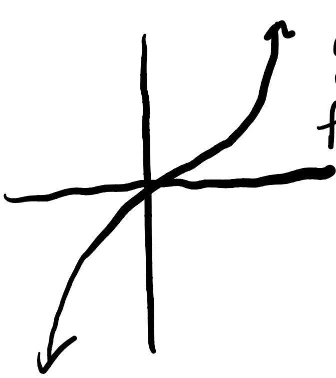
An even fcn has symmetry when reflected across the y-axis.



$y = x^2$ Note $f(-x) = (-x)^2 = (-x)(-x) = x^2 = f(x)$

$f(x) = x^2$

An odd fcn has symmetry when reflected across both axes or rotation of π about the origin:



$y = x^3$ Note $f(-x) = (-x)(-x)(-x)$

$f(x) = x^3$ $= -x^3 = -f(x)$.

Is it a coincidence that x^2 is even and x^3 is odd??

No, not a coincidence.

Example 2.3. (Challenging Exercise) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial of degree n. Show that

(a) if n is even, and $a_1, a_3, a_5, \ldots, a_{n-1} = 0$, then f is even.

(b) if n is odd, and $a_0, a_2, a_4, \ldots, a_{n-1} = 0$, then f is odd.

a) For any $x \in \mathbb{R}$:

$$f(-x) = a_n(-x)^n + a_{n-2}(-x)^{n-2} + \ldots + a_2(-x)^2 + a_0$$

But since $n, n-2, \ldots, 2$ are even numbers, we have (for example) $(-x)^n = \underbrace{(-1)^n}_{=+1} x^n = x^n$.

Then $f(-x) = a_n x^n + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_0 = f(x)$.

def'n of even!

$\therefore f$ is even.

b) For any $x \in \mathbb{R}$:

$$f(-x) = a_n(-x)^n + a_{n-2}(-x)^{n-2} + \ldots + a_3(-x)^3 + a_1(-x).$$

But since $n, n-2, \ldots, 3, 1$ are odd numbers, we have (for example) $(-x)^n = \underbrace{(-1)^n}_{=-1} x^n = -x^n$.

Then $f(-x) = -a_n x^n - a_{n-2}x^{n-2} - \ldots - a_3 x^3 - a_1 x = -f(x)$.

def'n of odd!

$\therefore f$ is odd.

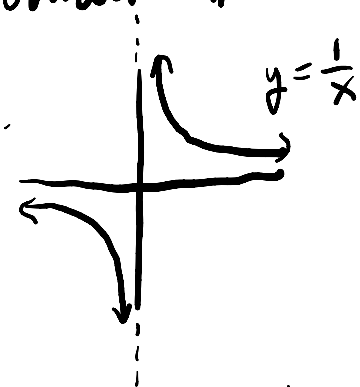
CATALOGUE OF COMMON FUNCTIONS, CT'D

(4) Rational functions

$$f(x) = \frac{P(x)}{Q(x)}$$ where P and Q are polynomials and $Q(x) \neq 0$.

Domain: $\mathbb{R}$ except for the roots of $Q(x)$.

e.g.



Domain is $\mathbb{R}$ except for $x=0$, so we can write
$(-\infty, 0) \cup (0, \infty)$
or $\{x \in \mathbb{R} \mid x \neq 0\}$

Note: $\frac{1}{x}$ is an odd fcn!

Example 2.4. Let $f(x) = \dfrac{-x^2 - 4x + 5}{x^2 - 1}$.

(a) Find the domain of f. Let's fully factor:

$$\frac{-x^2 - 4x + 5}{x^2 - 1} = \frac{-(x^2+4x-5)}{(x+1)(x-1)} = \frac{-(x+5)(x-1)}{(x+1)(x-1)}$$

there's a hole at $x=1$! Don't cancel this.

So $x=-1, 1$ are not in the domain
Domain: $\{x \in \mathbb{R} \mid x \neq \pm 1\}$

(b) What are the roots of f?

Ordinarily, we would just check where the numerator is 0: $x=-5$, $x=1$. But $x=1$ is not in the domain, so it cannot be a root.

The only root is $x=-5$.

Example 2.5. Find all solutions to $\dfrac{x+8}{x+1} > \dfrac{x}{x-5}$.

Move everything to the same side without multiplying/dividing:

$$\frac{x+8}{x+1} - \frac{x}{x-5} > 0.$$

$$\Rightarrow \frac{(x+8)(x-5) - x(x+1)}{(x+1)(x-5)} > 0$$

Expand & re-factor numerator:

$$\Rightarrow \frac{\cancel{x^2} + 3x - 40 - \cancel{x^2} - x}{(x+1)(x-5)} > 0$$

root @ $x = 20$

VAs @ $x = -1, 5$

$$\Rightarrow \frac{2x-40}{(x+1)(x-5)} > 0 \Rightarrow \frac{2(x-20)}{(x+1)(x-5)} > 0$$

Put roots and domain breaks in a table:

	$x<-1$	$-1<x<5$	$5<x<20$	$x>20$
$x-20$	−	−	−	+
$x+1$	−	+	+	+
$x-5$	−	−	+	+
	−	+	−	+

Inequality is satisfied when $-1<x<5$ or $x>20$

$(-1,5)\cup(20,\infty)$

or

$\{x\in\mathbb{R} \mid -1<x<5 \text{ or } x>20\}$

Practice Exercises

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