LEC 6 – DTDS Part I

We would like a mathematical model that reflects how some biological measurement changes over time. One way to do this is with a *Discrete-Time Dynamical System* (DTDS).

Definition 6.1. (DTDS) Suppose we have the following:

- a quantity x that can be measured at time t for any $t \in \mathbb{N}$ (we write x_t to mean "x at time t")
- a suitable time step that represents the amount of time between t and $t+1$;
- an updating fcn $f(x)$ that describes the change in x after one time step; i.e. $x_{t+1} = f(x_t)$

Then the DTDS that models this can be summarized by $x_{t+1} = f(x_t)$.

Example 6.2.

(1) (Medicine) A patient is administered 30 mcg of medicine every hour, and then the amount of medication present in the body is immediately measured. The drug has a half-life of 1 hour when it is in the body.

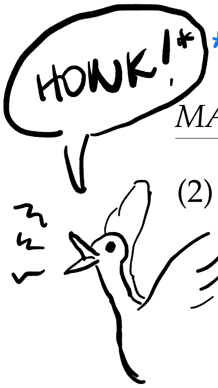
- M_t represents the amount of medicine in the body at time t.
- the time step is 1 hour.
- then the updating fcn is $f(x) = 0.5x + 30$

represents previous measurement

The DTDS is $M_{t+1} = 0.5 M_t + 30$

$f(x)$ x

same as updating fcn



*Translation: I WILL EAT YOU, HUMAN

(2) (Untitled Goose Example) The goose population on the front cover of the textbook is measured at the beginning of each school year. These positively *fowl* creatures hatch a number of goslings equal to 25% of the existing population, but over the course of the year, 80 geese perish.

- N_t represents goose population after time t
- time step is 1 year
- updating fcn is $f(x) = 1.25x - 80$

flock increases by 25% first

then 80 are lost

The DTDS is $N_{t+1} = 1.25N_t - 80$.

$N_0 = 50$

(3) (???) On a faraway island, there is a population of 50 pandas (animal). (Incidentally, a group this size is known as a depression (abstract noun) of pandas (animal).) It's common knowledge that Justin Bieber (celebrity)'s favourite hobby is ~~to~~ play basketball (activity) with them using a microwave (object). Unfortunately, this kills half the animals annually, so every two years in September, a helicopter delivers 40 pandas (animal) to the island one at a time.

- P_t, population at time t.

100 pandas
↓ $\cdot 0.5$
50
↓ $\cdot 0.5$
25

- time step of 1 year?
- updating fcn = ?
- $N_{t+2} = 0.5^2 N_t + 40$

2 years?

$f(x) = 0.25x + 40$

$N_{t+1} = 0.5^2 N_t + 40$

Definition 6.3. (Fixed Point, Equilibrium, Steady-state)
A point x^* of a DTDS $x_{t+1} = f(x_t)$ is called a <u>fixed point</u> (or <u>equilibrium</u>, or <u>steady-state</u>) if $f(x^*) = x^*$.

If $x_t = x^*$, then $x_{t+1} = f(x_t) = x_t$, which means x has not changed from one time step to the next.

Example 6.4.

(1) (Medicine) (Read example below first)

$$M_{t+1} = 0.5M_t + 30$$

Replace M_{t+1}, M_t with M^*:

$$M^* = 0.5M^* + 30$$

$$\Rightarrow M^* - 0.5M^* = 30$$

$$\Rightarrow 0.5M^* = 30$$

$$\Rightarrow M^* = 60$$

(2) (Geese) $N_{t+1} = 1.25N_t - 80$

Replace N_{t+1} and N_t with our desired fixed point, N^*, and solve for N^*.

$$N^* = 1.25N^* - 80$$

$$\Rightarrow N^* - 1.25N^* = -80$$

$$\Rightarrow -0.25N^* = -80$$

$$\Rightarrow N^* = 320$$

So a goose pop of 320 will be consistent from year to year.

What if the goose population is initially 400? How many geese will there be 3 years later?

We want to find N_3, starting with $N_0 = 400$.

$$N_1 = 1.25N_0 - 80 = 1.25 \cdot 400 - 80 = 420$$
$$N_2 = 1.25N_1 - 80 = 1.25 \cdot 420 - 80 = 445$$
$$N_3 = 1.25N_2 - 80 = 1.25 \cdot 445 - 80 = 556.25 - 80 = 476.25$$

round down to 476 for final answer

These geese are growing more and more each year. Perhaps it'll go up forever. I'm worried...

(3) (???) Let's try to come up with a formula for the fixed point of this *linear* DTDS.

Say $N_{t+2} = 0.25N_t + 40$

Replacing with N^* will work just as well.

$$N^* = 0.25N^* + 40$$
$$\Rightarrow N^* - 0.25N^* = 40$$
$$\Rightarrow N^*(1 - 0.25) = 40$$
$$\Rightarrow N^* = \frac{40}{1-0.25} = \frac{40}{0.75} \approx 53.3$$

If we wrote $N_{t+1} = rN_t + c$, then we ended up solving $N^* = \frac{c}{1-r}$.

This works for all <u>linear</u> updating fcns.

PRACTICE EXERCISES

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§3.1 pg. 126 # 1-4, 17-20

§3.2 pg. 139 # 23-30 (just find equilibria), 31-34