1. The exponential function $f(x) = Ca^x$ whose graph passes through points $(1, 4)$ and $(2, 32)$ is

   A. $f(x) = 4^x$

   B. $f(x) = 5^x$

   C. $f(x) = \frac{1}{2}\ 8^x$

   D. $32^{x/2}$

   E. none of the above

2. How many solutions does the equation $\sin(2x) = \cos x$ have in the interval $[0, 2\pi]$?

   A. none

   B. one

   C. two

   D. three

   E. more than three

3. Which of the following functions has an inverse:

   A. f(x)=3

   B. $g(x) = x^2$

   C. $h(x) = x^2 + 1$

   D. $k(x) = e^x$

   E. none of the above

4. If $f(x) = \ln(x + 5)$ then $f^{-1}(x) =$

   A. e^{x-5}

   B. $e^x - 5$

   C. $e^{x-\ln 5}$

   D. $\frac{1}{\ln(x-\frac{1}{5})}$

   E. none of the above

5. (Remember no calculators or electronic aids may be used in the exam)
   $\log_3 \frac{1}{27} + \log_3 30 - \log_3 10 =$

   A. -3

   B. 3

   C. -2

   D. 2

   E. none of the above

6. $\tan(\arctan 8) =$

   A. -8

   B. 8

   C. $\pi - 8$

   D. $\frac{\pi}{2} - 8$

   E. none of the above

7. $\cos(2\tan^{-1} x) =$

A. $\frac{1+x^2}{1-x^2}$

B. $\frac{1-x^2}{1+x^2}$

C. $\frac{x}{1+x^2}$

D. $\frac{x}{1-x^2}$

E. none of the above

8. $\lim_{x\to 4^+} \ln(x^2-16) =$

A. 0

B. 1

C. $-\infty$

D. ∞

E. none of the above

9. $\lim_{h\to 0} \frac{(3h+1)^2-1}{h} =$

A. 0

B. 6

C. 9

D. 15

E. none of the above

10. $\lim_{x\to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2} =$

A. 0

B. $-\frac{1}{4}$

C. $-\frac{1}{2}$

D. -1

E. none of the above

11. $\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} =$

A. -1

B. 0

C. 1

D. 2

E. none of the above

12. $\lim_{x\to\infty} \arctan(e^{2x}) =$

A. $\frac{\pi}{2}$

B. 0

C. ∞

D. $-\infty$

E. none of the above

13. $\lim_{x\to-\infty} \frac{x^{1/3}+x^2}{1+x+2x^2} =$

A. $-\frac{1}{2}$

B. 0

C. $\frac{1}{2}$

D. 1

E. none of the above

14. For the graph of the function $y = \frac{2x^2+x-1}{x^2+x-2}$ which of the following statements is true ?

A. This graph has two vertical asymptotes and one horizontal asymptote

B. x=3 is a vertical asymptote

C. y=3 and y=0 are horizontal asymptotes

D. there are no vertical asymptotes

E. statements A,B,C,D are false

15. Equality $\lim_{h\to0} \frac{\sin(\frac{\pi}{2}+h)-1}{h} = f'(a)$ holds for

A. $f(x) = \sin x,\ a = 0$

B. $f(x) = \sin x,\ a = \frac{\pi}{2}$

C. $f(x) = \sin(\frac{\pi}{2} + x),\ a = \frac{\pi}{2}$

D. $f(x) = \sin(\frac{\pi}{2} + x) - 1,\ a = 0$

E. none of the above

16. $\frac{d}{dx}e^5 =$

A. 0

B. e^5

C. $5e^4$

D. $\ln 5$

E. none of the above

17. The slope of the tangent line to the graph of $y = \sin x - \cos x$ at $(\pi, 1)$ is

A. 2

B. 1

C. 0

D. -1

E. none of the above

18. If $xe^y = x - y$ then the value of $\frac{dy}{dx}$ when $x = 1$ and $y = 0$ is

A. 0

B. $\frac{1}{2}$

C. 1

D. 2

E. none of the above

19. If $f(x) = \ln(\ln(x^2 + 1))$ then $f'(1) =$

A. $\ln(\ln 2)$

B. $\frac{2}{\ln 2}$

C. $\frac{1}{\ln 2}$

D. $\frac{1}{2\ln 2}$

E. none of the above

20. $\frac{d}{dx}\cos(e^x + x) =$

A. $\sin(e^x + x)$

B. $-\sin(e^x + x)$

C. $(e^x + 1)\sin(e^x + x)$

D. $-(e^x + 1)\sin(e^x + x)$

E. none of the above

21. $\lim_{x\to -3} \frac{x+3}{x^2-9} =$

A. $\frac{1}{6}$

B. $-\frac{1}{6}$

C. ∞

D. 0

E. none of the above

22. An equation of the tangent line to $y = 3e^{x-1}$ at $(1, 3)$ is

A. $y = 3x$

B. $y = \frac{3}{e}x$

C. $y = 3x - 1$

D. $y = 3x - e$

E. none of the above

23. If $g(x) = \sin(f(x) + \pi)$, $f(\pi) = 0$, and $f'(\pi) = 2$, then $g'(\pi) =$

A. π

B. 2

C. 0

D. -2

E. none of the above

In the questions that follow you are required to show your work. Questions without sufficient justification will not receive credit.

28. Let

$$f(x) = \begin{cases} \frac{x^2+x-6}{|x-2|} & \text{if } x \neq 2 \\ 5 & if\ x = 2 \end{cases}$$

(i) Find $\lim_{x\to 2-} f(x)$

$x \to 2^-$ means we only consider $x < 2$

$x^2+x-6 = (x-2)(x+3)$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2+x-6}{|x-2|}$

$= \lim_{x\to 2^-} \frac{x-2}{|x-2|}(x+3) = \lim_{x\to 2^-} \frac{x-2}{|x-2|} \cdot \lim_{x\to 2^-} x+3$ (product law)

$= 5 \cdot \lim_{x\to 2^-} \frac{x-2}{|x-2|}$ If $x<2$ $x-2<0$

$|x-2| = -(x-2)$

$= 5 \cdot \lim_{x\to 2^-} \frac{\cancel{x-2}}{-(\cancel{x-2})} = 5\cdot(-1) = \boxed{-5}$

(ii) Find $\lim_{x\to 2+} f(x)$

$x \to 2^+$ means $x > 2$ $\to$ $x-2>0$ so $|x-2| = x-2$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^2+x-6}{|x-2|} = \lim_{x\to 2^+} \frac{\cancel{(x-2)}(x+3)}{\cancel{x-2}} = \lim_{x\to 2^+} x+3$

$= \boxed{5}$

(iii) Find $\lim_{x\to 2} f(x)$

Two-sided limits exist only if the one sided limits agree.

$\lim_{x\to 2^-} f(x) = -5 \neq 5 = \lim_{x\to 2^+} f(x)$

$\therefore \lim_{x\to 2} f(x)$ DNE

(iv) Is $f(x)$ continuous at $x = 2$? Why ?

To be cts at $x=2$ $f(2)$, $\lim_{x\to 2} f(x)$ to exist and be equal. But by (iii) $\lim_{x\to 2} f(x)$ DNE.

So f is not cts at $x=2$.

(v) Is $f(x)$ continuous at $x = 0$? Why ?

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x^2+x-6}{|x-2|} = \lim_{x\to 0^-} \frac{(x-2)(x+3)}{|x-2|}$$

for $x<0$ $\quad x-2<0$ $\quad$ so $|x-2| = -(x-2)$

$$= \lim_{x\to 0^-} \frac{\cancel{(x-2)}(x+3)}{-\cancel{(x-2)}} = \lim_{x\to 0^-} -(x+3) = -3$$

for $\underline{x>0}$ $\underline{\text{eventually}}$ the values of x we consider are $\underline{\text{less than } 2}$. Then $\underline{x-2<0}$

so $|x-2| = -(x-2)$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2+x-6}{|x-2|} = \lim_{x\to 0^+} \frac{(x-2)(x+3)}{-(x-2)} = \lim_{x\to 0^+} -(x+3)$$

$= -3$

$$\lim_{x\to 0^-} f(x) = -3 = \lim_{x\to 0^+} f(x) \quad \therefore \lim_{x\to 0} f(x) = -3 \checkmark$$

$$f(0) = \frac{0^2+0-6}{|0-2|} = \frac{-6}{|-2|} = \frac{-6}{2} = -3$$

$$\boxed{f(0) = \lim_{x\to 0} f(x)} \quad \therefore f \text{ is cts at } x=0$$

29. Use the Intermediate Value Theorem to show that the equation

$$4^x = 5 - 3x$$

has a solution in the interval $(0, 1)$.

$f(x)$ • cts $[a,b]$ ✓

D between $f(a), f(b)$

there is d in (a,b)

so $f(d) = D$

$$4^x + 3x - 5 = 0$$

Let $f(x) = 4^x + 3x - 5$

f is continuous.

$\therefore$ f is cts on $[0,1]$

Let $D = 0$

$f(0) = 4^0 + 3(0) - 5 = 1 + 0 - 5 = -4$

$f(1) = 4^1 + 3(1) - 5 = 4 + 3 - 5 = 2$

Since 0 lies strictly between $f(0)$ and $f(1)$ the IVT tells us there is at least one x in $(0,1)$ so that $f(x) = 0$

$$f(x) = 0$$

$$4^x + 3x - 5 = 0$$

$$4^x = 5 - 3x$$

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

cos -sin

$x^2 + \arctan(e^x \sin(x))$

30. For
$$f(x) = \cos(x^2 + \arctan(e^x \sin x))$$
find $f'(x)$.

$$f(x) = \cos\left(x^2 + \arctan(e^x \sin(x))\right)$$

$$f'(x) = \left[\cos\left(x^2 + \arctan(e^x \sin(x))\right)\right]'$$

chain rule

$$= -\sin\left(x^2 + \arctan(e^x \sin(x))\right) \cdot \left[x^2 + \arctan(e^x \sin(x))\right]'$$

sum rule

$$\left[x^2 + \arctan(e^x \sin(x))\right]' = [x^2]' + \left[\arctan(e^x \sin(x))\right]'$$

$$= 2x + \left[\arctan(e^x \sin(x))\right]'$$

chain rule

$$\left[\arctan(e^x \sin(x))\right]' = \frac{1}{(e^x \sin(x))^2 + 1} \cdot \left[e^x \sin(x)\right]'$$

$\arctan(x)$ $\qquad \frac{1}{x^2+1}$

$e^x \sin(x)$

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

product rule

$$\left[e^x \cdot \sin(x)\right]' = e^x \sin(x) + e^x \cos(x)$$

Putting our work together

$$\boxed{f'(x) = -\sin\left(x^2 + \arctan(e^x \sin(x))\right) \cdot \left(2x + \frac{e^x \sin(x) + e^x \cos(x)}{(e^x \sin(x))^2 + 1}\right)}$$

31. If $F(x) = f(g(x))$ where $f(\pi) = 8$, $f'(\pi) = 4$, $f'(6) = 3$, $g(6) = \pi$, and $g'(6) = 7$, find $F'(6)$.

$$F'(x) \overset{\text{chain rule}}{=} f'(g(x)) \cdot g'(x)$$

$$F'(6) = f'(g(6)) \cdot g'(6)$$

$$= f'(\pi) \cdot 7$$

$$= 4 \cdot 7 = \boxed{28}$$

32. (a) If $x^2 - xy + y^2 = 3$, use implicit differentiation to find $\frac{dy}{dx}$ when $x = -1$, $y = 1$.

$$[x^2 - xy + y^2]' = [3]'$$

derive both sides.

$$[x^2]' - [xy]' + [y^2]' = 0$$

product rule

chain rule

$$2x - \left(x\frac{dy}{dx} + 1\cdot y\right) + \left(2y\frac{dy}{dx}\right) = 0$$

$$2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$$

$$(2y - x)\frac{dy}{dx} = y - 2x$$

$$\boxed{\frac{dy}{dx} = \frac{y-2x}{2y-x}}$$

At $x = -1$, $y = 1$

$$\frac{dy}{dx} = \frac{1 - 2(-1)}{2(1) - (-1)} = \frac{1+2}{2+1} = \frac{3}{3} = 1$$

(b) Use your answer in part (a) to find an equation of the tangent line to the curve $x^2 - xy + y^2 = 3$ at $(-1, 1)$.

From (a) we know at $(-1, 1)$ $\frac{dy}{dx} = 1$

$\frac{dy}{dx}$ is the slope of the tangent at $(-1, 1)$

Point slope form: $y - 1 = 1(x - -1)$

$$y - 1 = x + 1$$

$y = mx + b$ form! (y intercept)

$$\boxed{y = x + 2}$$