MIDTERM SOLUTION

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Exercise 1. (1) *An open rectangular box (no top) with volume* 5 *cubic meters has a square base. If the length of each side of the base is x and the height is h, express the surface area S of the box as function of x only (*<u>*not*</u> *of x and h).*

(2) *Solve for x (find the* exact *values, do not approximate):*

$$2\log_4(x) - \log_4(x^2-3) = 1.$$

(3) *Find the inverse function $f^{-1}(x)$ of $f(x) = \ln(3x+1)$ and determine the domain and range of $f^{-1}(x)$.*

<u>Solution</u>:

(1) Let us first express the volume V of the cubic box in terms of x and h:

$$V = x^2 \cdot h = 5\ cm^3. \tag{0.1}$$

The surface area A depends also on h and x:

$$A = 4x \cdot h + x^2. \tag{0.2}$$

However, the equality in (0.1) gives us a description of h in terms of x:

$$h = \frac{5}{x^2}.$$

By substituting this expression of h in the equality (0.2), we obtain a description of the surface in terms of x only:

$$A = 4x \cdot \frac{5}{x^2} + x^2 = \frac{20}{x} + x^2.$$

(2) We are going to us the following laws related to logarithms for any base b

$$a\log_b(x) = \log_b(x^a), \tag{0.3}$$

$$\log_b(x) - \log_b(y) = \log_b(\frac{x}{y}), \tag{0.4}$$

$$b^{\log_b(x)} = x. \tag{0.5}$$

$$\begin{aligned}
2\log_4(x) - \log_4(x^2-3) &= 1 \\
\log_4(x^2) - \log_4(x^2-3) &= 1 && \text{By (0.3) we have indeed } 2\log_4(x) = \log_4(x^2) \\
\log_4(\frac{x^2}{x^2-3}) &= 1 && \text{By (0.4) we have } \log_4(x^2) - \log_4(x^2-3) = \log_4(\frac{x^2}{x^2-3}) \\
\frac{x^2}{x^2-3} &= 4^1 && \text{By (0.5) we take both sides as exponents with base 4} \\
x^2 &= 4(x^2-3)
\end{aligned}$$

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The solutions to this last equation are $x_1 = 2$ and $x_2 = -2$. However, we discard $x_2 = -2$, since the logarithm may not have negative argument. So, the only solution is $x = 2$.

(3) We set $y = \ln(3x+1)$ and solve the equation for x:

$$\begin{aligned} y &= \ln(3x+1) \\ e^y &= e^{\ln(3x+1)} && \text{We take both sides of the equation as exponents with base } e \\ e^y &= 3x+1 && \text{Here we use that } e^{\ln(x)} = x \\ e^y - 1 &= 3x \\ \frac{e^y-1}{3} &= x \end{aligned}$$

We then switch the variables x and y and obtain that

$$f^{-1}(x) = \frac{e^x - 1}{3}.$$

Let us now determine domain and range of f^{-1}. Recall that given an invertible function f, we have that

$$\begin{aligned} \text{domain of } f^{-1}(x) &= \text{ range of } f(x), \\ \text{range of } f^{-1}(x) &= \text{ domain of } f(x) \end{aligned}$$

The range of $f(x)$ is $(-\infty, +\infty)$, since f is a logarithmic function, hence the domain of $f^{-1}(x)$ is $(-\infty, +\infty)$. On the other hand, the domain of $f(x) = \ln(3x+1)$ is given by the condition $3x + 1 > 0$, that is $x > -1/3$, hence the range of f^{-1} is $(-1/3, +\infty)$.

Exercise 2. *Find the limit or explain why the limit does not exist:*

(1)

$$\lim_{x \to 3} \frac{\sqrt{x^2+7} - 4}{x^2 - 9}$$

(2)

$$\lim_{x \to -2} \frac{|x+2|}{x^2 - 4}$$

<u>Solution</u>:

(1) If we were to use the limit laws right away, we would obtain an indeterminate form:

$$\lim_{x \to 3} \frac{\sqrt{x^2+7} - 4}{x^2 - 9} = \frac{'0'}{0}$$

We need therefore to re-write suitably the function whose limit we want to compute. Since a square root appears, we will use the completion of squares

$$(a-b)(a+b) = a^2 - b^2,$$

with $a = \sqrt{x^2+7}$ and $b = 4$ in order to get rid of the square root on the top.

$$\lim_{x\to 3} \frac{\sqrt{x^2+7}-4}{x^2-9} = \lim_{x\to 3} \frac{\sqrt{x^2+7}-4}{x^2-9} \cdot \frac{\sqrt{x^2+7}+4}{\sqrt{x^2+7}+4} \qquad \text{(the equality holds since } \frac{\sqrt{x^2+7}+4}{\sqrt{x^2+7}+4} = 1\text{)}$$

$$= \lim_{x\to 3} \frac{x^2+7-16}{(x^2-9)\cdot(\sqrt{x^2+7}+4)}$$

$$= \lim_{x\to 3} \frac{x^2-9}{(x^2-9)\cdot(\sqrt{x^2+7}+4)}$$

$$= \lim_{x\to 3} \frac{1}{\sqrt{x^2+7}+4} = \frac{1}{8}.$$

The limit therefore exists and is equal to 1/8.

(2) In order to compute any limit containing an absolute value, we need to take care of the absolute value and make a distinction. Remark that by definition of absolute value, for any function $f(x)$ we have

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

In this case, we therefore have that

$$|x+2| = \begin{cases} x+2 & \text{if } x+2 \geq 0, \text{ hence } x \geq -2 \\ -x-2 & \text{if } x+2 < 0, \text{ hence } x < -2 \end{cases}$$

This tells us that we have to make a distinction between *limit from the left* and *limit from the right*, because according to the direction we will be dealing with slightly different functions (that is, with different signs). Let us start by taking the limit from the right

$$\lim_{x\to -2^+} \frac{|x+2|}{x^2-4} = lim_{x\to -2+} \frac{x+2}{x^2-4} \qquad \text{the equality holds since } x \to -2^+, \text{ hence } x > -2$$

$$= \lim_{x\to -2^+} \frac{x+2}{(x+2)(x-2)} = \frac{1}{x-2} = -\frac{1}{4}.$$

On the other hand, the limit from the left is

$$\lim_{x\to -2^-} \frac{|x+2|}{x^2-4} = lim_{x\to -2-} \frac{-x-2}{x^2-4} \qquad \text{the equality holds since } x \to -2^-, \text{ hence } x < -2$$

$$= \lim_{x\to -2^-} \frac{-x-2}{(x+2)(x-2)} = \frac{-1}{x-2} = \frac{1}{4}.$$

Therefore limits from the right and from the left don't coincide

$$\lim_{x\to -2^+} \frac{|x+2|}{x^2-4} \neq \lim_{x\to -2^-} \frac{|x+2|}{x^2-4}$$

and therefore the limit does not exist.

Exercise 3. *Find the first and second derivatives of the function*

$$f(x) = x2^x(1+2^{-x}).$$

Solution: Let us first simplify the function as suggested in the Hint[1]

$$f(x) = x2^x(1 + 2^{-x}) = x2^x + x2^x 2^{-x} = x2^x + x2^{x-x} = x2^x + x2^0 = x2^x + x.$$

- *First derivative:* In order to derive this function, we need to use the sum and the product rules:

$$\begin{aligned} f'(x) &= (x2^x)' + (x)' && \text{By the sum rule} \\ &= (x)'2^x + x(2^x)' + (x)' && \text{By the product rule} \end{aligned}$$

  therefore all we need is to determine the derivative $(2^x)'$. One should remember that the derivative of an exponential function is proportional to the function itself. One does not need to remember such derivative by heart, it can be obtained easily by using the following trick: by the logarithmic and exponential laws, we have that

$$2^x = e^{\ln(2^x)} = e^{x\ln(2)},$$

  and therefore

$$(2^x)' = (e^{x\ln(2)})' = \ln(2)e^{x\ln(2)} = \ln(2)2^x.$$

  From this, we obtain that

$$f'(x) = 2^x + x\ln(2)2^x + 1.$$

- *Second derivative:* Recall that the second derivative is the derivative of the derivative, hence

$$f''(x) = (f'(x))' = \ln(2)2^x + \ln(2)2^x + x\ln(2)\ln(2)2^x.$$

Exercise 4. *Find all horizontal and all vertical asymptotes of the graph*

$$y = \frac{\sqrt{4x^6 - 9x^2 + 1}}{x^3 + 5x}.$$

Soultion:

- *Vertical asymptotes:* Following the definition, a vertical line $x = a$ is called a *vertical asymptote* of a curve $y = f(x)$ if at least one of the following statements are true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

  In order to have one of these situations in the case of a quotient function (like the one in the exercise), we need to check for which values of x the quotient of the function is equal to 0, hence we verify for which x we have

$$x^3 + 5x = 0$$

  . We factor this equation as

$$x(x^2 + 5) = 0$$

[1]Remark: This is just a suggestion. If you don't feel comfortable simplifying the function, you don't need to do so. However, finding a correct simplification of the function makes the exercise easier.

and we obtain that this equation is equal to zero when at least one of the factors is equal to zero, that is

$$x = 0 \text{ or } x^2 + 5 = 0.$$

Remark that the second is not possible, as $x^2 + 5 = 0$ if and only if $x^2 = -5$ and this is a negative square, which does not make sense. Hence the only vertical asymptote of the function $f(x)$ is the line $x = 0$.

- *Horizontal asymptotes:* According to the definition, a horizontal line $y = L$ is called a *horizontal asymptote* of the curve $y = f(x)$ if one of the following conditions are satisfied:

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L.$$

Therefore, in order to find the horizontal asymptotes, we need to compute both $\lim_{x\to\infty} \frac{\sqrt{4x^6-9x^2+1}}{x^3+5x}$ and $\lim_{x\to-\infty} \frac{\sqrt{4x^6-9x^2+1}}{x^3+5x}$.

$$\lim_{x\to\infty} \frac{\sqrt{4x^6-9x^2+1}}{x^3+5x} = \lim_{x\to\infty} \frac{\sqrt{x^6(4-9\frac{x^2}{x^6}+\frac{1}{x^6})}}{x^3(1+5\frac{x}{x^3})} \quad \text{We want to verify which between the num. and the denim. goes faster to } \infty$$

$$= \lim_{x\to\infty} \frac{\sqrt{4}x^3}{x+3} = 2 \quad \text{We take +2 as the root of 4, since the function } f(x) \text{ is positive as } x \text{ approaches } +\infty$$

The computation for the limit as $x \to -\infty$ is very similar:

$$\lim_{x\to-\infty} \frac{\sqrt{4x^6-9x^2+1}}{x^3+5x} = \lim_{x\to-\infty} \frac{\sqrt{x^6(4-9\frac{x^2}{x^6}+\frac{1}{x^6})}}{x^3(1+5\frac{x}{x^3})}$$

$$= -2 \quad \text{We take } -2 \text{ as the root of 4, since the function } f(x) \text{ is negative as } x \text{ approaches } -\infty$$

Hence there are two horizontal asymptotes: $y = 2$ and $y = -2$.

Exercise 5. *Find the derivatives of the following functions*

(1) $f(x) = (x + 2\sqrt{x})\tan(x) + \frac{1}{x^2}$,
(2) $f(x) = e^{-2x}\sin^2(x)$,
(3) $f(x) = \frac{\cos^2 x}{1+\tan x}$,
(4) $f(x) = \sin[x^2 + \cos(x\sin(x))]$.

Solution:

(1) By the sum and the product rules, we obtain

$$f'(x) = (x + 2\sqrt{x})'\tan(x) + (x + 2\sqrt{x})(\tan(x))' + (\frac{1}{x^2})'$$
$$= (1 + 2\frac{1}{2\sqrt{x}})\tan(x) + (x + 2\sqrt{x})\sec^2(x) + \frac{-2}{x^3}.$$

(2)

$$f'(x) = (e^{-2x})'\sin^2(x) + e^{-2x}(\sin^2(x))' \qquad \text{This is obtained by the product rule}$$
$$= (-2e^{-2x})\sin^2(x) + e^{-2x}(2\sin(x)\cos(x)).$$

In the last line we used the chain rule, as both e^{-2x} and $\sin^2(x)$ are composite functions. Remark in particular that $\sin^2(x) = (\sin(x))^2$.

(3) We are going to use the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(\cos^2(x))'(1+\tan(x)) - \cos^2(x)(1+\tan(x))'}{(1+\tan(x))^2} \\ &= \frac{(-2\cos(x)\sin(x))(1+\tan(x)) - \cos^2(x)(\sec^2(x))}{(1+\tan(x))^2} \qquad \text{Remark that } \cos^2(x) = (\cos(x))^2 \\ &= \frac{(-2\cos(x)\sin(x))(1+\tan(x)) - 1}{(1+\tan(x))^2} \end{aligned}$$

(4) This is a composition of function and we will therefore use the chain rule:

$$\begin{aligned} f'(x) &= \cos[x^2 + \cos(x\sin(x))](x^2 + \cos(x\sin(x)))' \\ &= \cos[x^2 + \cos(x\sin(x))](2x + \sin(x\sin(x))(\sin(x) + x\cos(x))) \end{aligned}$$

Exercise 6. *Given the function* $f(x) = \frac{5x}{1+x^2}$, *write equation of the tangent line to the curve* $y = f(x)$ *at the point* $(2, f(2))$.

Solution: A line has equation

$$y = mx + q, \tag{0.6}$$

where m denotes its slope and q denotes its intersection with the y-axis. The slope of the tangent line to the curve $y = f(x)$ at 2 can be found by evaluating $f'(2)$. By the quotient rule we have that

$$f'(x) = \frac{5(1+x^2) - 5x(2x)}{(1+x^2)^2} = \frac{-5x^2+5}{x^4+2x^2+1},$$

and hence

$$f'(2) = -\frac{3}{5}.$$

Moreover we have that

$$f(2) = \frac{10}{5} = 2.$$

Therefore we use all the data to find q starting from equation (0.6):

$$\begin{aligned} f(2) &= f'(2)(2) + q \\ 2 &= -\frac{3}{5}\cdot 2 + q \\ 2 + \frac{6}{5} &= q \\ \frac{16}{5} &= q \end{aligned}$$

We obtain therefore that the tangent line has equation

$$y = -\frac{3}{5}x + \frac{16}{5}.$$

Exercise 7 (Bonus question)**.** *Recall that* f *is called an even function if* $f(-x) = f(x)$ *and* f *is an odd function if* $f(-x) = -f(x)$. *Use the chain rule to prove that for any function* $g(x)$ *which is differentiable at all non-negative* x, *the derivative* $f'(x)$ *of the composite function* $f(x) = g(x^2)$.

Solution: Let us apply the chain rule to the function f:

$$f'(x) = (g(x^2))' = g'(x^2) \cdot 2x.$$

Hence we obtain that f' satisfies the definition of an odd function, indeed:

$$f'(-x) = g'((-x)^2) \cdot 2(-x) = g'(x^2) \cdot (-2x) = -g'(x^2) \cdot 2x = -f'(x).$$