1. $f(x,y) = x^3 - x^2y + y^2 - x^2$

$$f_x = 3x^2 - 2xy - 2x = 0 \quad \Big\} \quad x(3x - 2y - 2) = 0$$

$$f_y = -x^2 + 2y = 0 \qquad\qquad y = \frac{x^2}{2}$$

$\Rightarrow x = 0$ ou $3x - 2y - 2 = 0$

If $x = 0$, $y = \frac{0^2}{2} = 0 \Rightarrow (0,0)$

If $3x - 2y - 2 = 0 \Rightarrow 2y = 3x - 2$

$y = \frac{3x-1}{2}$

$\Rightarrow \frac{3x}{2} - 1 = \frac{x^2}{2} \Rightarrow 3x - 2 = x^2$

$\Rightarrow x^2 - 3x + 2 = 0$

$\Rightarrow (x-1)(x-2) = 0 \Rightarrow x = 1$ ou $x = 2$

si $x = 1$, $y = 1/2$ $\quad (1, 1/2)$

si $x = 2$, $y = 2$ $\quad (2,2)$

$f_{xx} = 6x - 2y - 2 \qquad f_{xy} = -2x$

$f_{yy} = 2$

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = 2(6x - 2y - 2) - (-2x)^2$

$= 12x - 4y - 4 - 4x^2$

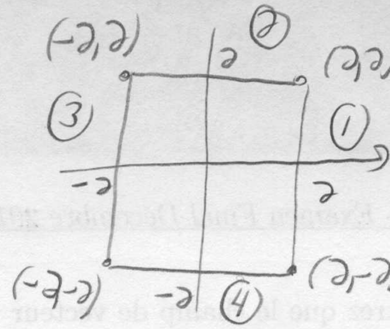
$D(0,0) = -4$ hence, saddle-point

$D(1, 1/2) = 12 - 2 - 4 - 4 = 2 > 0$ hence local minimum

$f_{xx}(1, 1/2) = 6 - 1 - 2 = 3 > 0$

$D(2,2) = 24 - 8 - 4 - 16 = -4 < 0$, hence saddle-point

2. $f(x,y) = x^2+y^2+x^2y-3$



$f_x = 2x + 2xy = 0 \quad 2x(1+y) = 0$

$f_y = 2y + x^2 = 0 \quad 2y + x^2 = 0$

$2x(1+y) = 0$

$\Rightarrow x = 0$ or $y = -1$

If $x=0,\ y=0 \Rightarrow \boxed{(0,0)}$

If $y=-1,\ x^2-2=0 \Rightarrow x = \pm\sqrt{2} \Rightarrow \boxed{(\sqrt{2},-1),\ (-\sqrt{2},-1)}$

$\boxed{f(0,0) = -3}$

$\boxed{f(\sqrt{2},-1) = 2+1-2-3 = -2}$

$\boxed{f(-\sqrt{2},-1) = 2+1-2-3 = -2}$

(1) $x=2,\ -2 \le y \le 2$

$f(2,y) = 4+y^2+4y-3 = y^2+4y+1 = f_1(y)$

$f_1'(y) = 2y+4 = 0 \Rightarrow y = -2$

$f(2,-2) = 4+4-8-3 = -3$

$f(2,2) = 4+4+8-3 = 13$

(2) $y=2,\ -2 \le x \le 2$

$f(x,2) = x^2+4+2x^2-3 = 3x^2+1 = f_2(x)$

$f_2'(x) = 6x = 0$ when $x=0$ $\quad f(0,2) = 4-3 = 1$

$f(-2,2) = 4+4+8-3 = 13 \quad f(2,2) = 13$

(3) $x=-2, \quad -2\le y\le 2$

$f(-2,y) = 4+y^2+4y-3 = y^2+4y+1 = f_3(y)$

$f_3'(y) = 2y+4 = 0 \Rightarrow y=-2$

Hence, $f(-2,-2) = 4+4-8-3 = -3$

$f(-2,2) = 13$

(4) $y=-2, \quad -2\le x\le 2$

$f(x,-2) = x^2+4-2x^2-3 = -x^2+1 = f_4(x)$

$f_4'(x) = -2x = 0 \Rightarrow x=0$

$\boxed{f(0,-2) = 1}$ $\quad \boxed{f(2,-2) = -3}$

$\boxed{f(-2,-2) = -3}$

maximum : 13

minimum : -3

3: $f(x,y) = x + y^2$ $\qquad x^2+y^2=1$

$\nabla f = (1, 2y) \qquad \nabla g = (2x, 2y)$

$\nabla f = \lambda \nabla g$

$$\left.\begin{array}{l}(1,2y) = \lambda(2x,2y)\\ x^2+y^2=1\end{array}\right\} \Rightarrow \boxed{\begin{array}{l}2x\lambda = 1\\ 2\lambda y = 2y\\ x^2+y^2=1\end{array}}$$

$2\lambda y = 2y \Rightarrow 2\lambda y - 2y = 0 \Rightarrow y = 0$ or $\lambda = 1$

$2y(\lambda - 1) = 0$

If $\underline{y=0}$: $x^2 + 0^2 = 1$

$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$\Rightarrow \boxed{(1,0) \text{ or } (-1,0)}$

$\boxed{f(1,0) = 1 \quad , \ f(-1,0) = -1}$

$\underline{\text{if } \lambda = 1}$: $2x = 1 \Rightarrow x = 1/2$ $\qquad x^2+y^2=1$

$2y = 2y$ $\qquad$ becomes

$\frac{1}{4} + y^2 = 1$

$\Rightarrow y^2 = 3/4$

$\Rightarrow (1/2, \sqrt{3}/2)$ or $(1/2, -\sqrt{3}/2)$ $\qquad \Rightarrow y = \frac{\pm\sqrt{3}}{2}$

$\boxed{f(1/2, \sqrt{3}/2) = 1/2 + 3/4 = 5/4}$

$\boxed{f(1/2, -\sqrt{3}/2) = 1/2 + 3/4 = 5/4}$

$$\boxed{\begin{array}{l}\underline{\text{maximum}}:\ 5/4\\ \underline{\text{minimum}}:\ -1\end{array}}$$

4. $f(x,y) = xye^{-(x^2+y^2)}$

$f_x = ye^{-(x^2+y^2)} + xye^{-(x^2+y^2)}(-2x) = ye^{-(x^2+y^2)}(1-2x^2) = 0$

$f_y = xe^{-(x^2+y^2)} + xye^{-(x^2+y^2)}(-2y) = xe^{-(x^2+y^2)}(1-2y^2) = 0$

$ye^{-(x^2+y^2)}(1-2x^2) = 0 \Rightarrow y = 0$ or $1-2x^2 = 0$

$\left(e^{-(x^2+y^2)} > 0\right)$ note that

$xe^{-(x^2+y^2)}(1-2y^2) = 0 \Rightarrow x = 0$ or $1-2y^2 = 0$

If $y=0$: Either $x=0$ or $1-2y^2=0$

but $y=0$ and $1-2y^2 = 1 = 0$ is impossible

Hence, $x=0$ with gives $(0,0)$ as a critical point.

Si $1-2x^2 = 0 \Rightarrow x^2 = 1/2 \Rightarrow x = \pm 1/\sqrt{2}$

the second equation gives $x=0$ (qui est impossible)

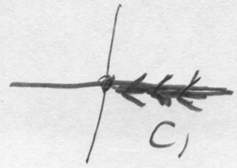
or $1-2y^2=0$

$\Rightarrow y = \pm 1/\sqrt{2}$.

Hence, the critical points are

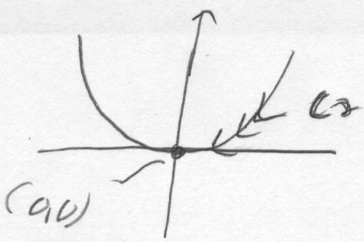
$\boxed{(1/\sqrt{2}, 1/\sqrt{2}),\ (1/\sqrt{2}, -1/\sqrt{2}),\ (-1/\sqrt{2}, 1/\sqrt{2}),\ (-1/\sqrt{2}, -1/\sqrt{2}),\ (0,0).}$

5a) C_1: $y=0$
$x>0$ and $x\to 0$



$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0}\frac{0}{x^4} = \lim_{x\to 0} 0 = 0$$

C_2: $y=x^2$, $x>0$, $x\to 0$



$$\lim_{\substack{(x,y)\to(0,0)\\ C_2}} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0}\frac{x^2x^2}{x^4+x^4}$$

$$= \lim_{x\to 0}\frac{x^4}{2x^4} = \lim_{x\to 0} \tfrac{1}{2} = \tfrac{1}{2}$$

not the same limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$ does not exist

b) $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = 2x(-\sin t+\cos t) + 2y(-\sin t-\cos t)$$

$$\left.\frac{dz}{dt}\right|_{t=0} = 2(1)(-0+1) + 2(1)(-0-1)$$
$$= 2(1)(1) + 2(1)(-1) = 2-2 = 0.$$

c) $Df = (2y,\ 2x-6y)$ $\quad Df(5,5) = (10,-20)$

$u = \frac{(4,3)}{\sqrt{16+9}} = (4/5,\ 3/5)$ $\quad D_uf(5,5) = (10,-20)\cdot(4/5,\ 3/5)$
$$= \frac{40}{5} - \frac{60}{5} = 8-12 = -4$$

d) $Dg(x,y,z) = (2x-y,\ -x-2y,\ -1)$.

$g(x,y,z) = x^2-xy-y^2-z$

$n = Dg(1,1,-1) = (1,-3,-1)$

$n\cdot(x-1,y-1,z+1) = 0 \Rightarrow (1,-3,-1)\cdot(x-1,y-1,z+1) = 0$

$(x-1)-3(y-1)-(z+1) = 0$

$x-3y-z+1 = 0 \Rightarrow \boxed{x-3y-z=-1}$