Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

MAT 1341D – The Midterm Test I (v1) **Instructor: Kirill Zaynullin**

Last name: ____________________

First name: ____________________

DGD group Nr: ____________________

Student number:

Please, read the following instructions carefully:

- You have 80 minutes to complete this test. **Do not detach** the pages of this booklet. Read each question carefully. Answer all questions by choosing (crossing) the respective box. Where it is possible to check your work, do so.
- This is a closed book exam, and no notes of any kind are allowed. The use of cell phones, laptops or any text storage or communication devices is not permitted.
- The use of calculators with a SIN button is not permitted on the test.

THIS SPACE IS RESERVED FOR THE MARKER:

Question	1	2	3	4	5	6	7	8	Total
Mark									
Out of	1	1	1	1	5	4	1	1	15

1. The volume of the parallelepiped with edges given by the vectors $\vec{u} = (1,1,0)$, $\vec{v} = (1,3,2)$ and $\vec{w} = (1,1,3)$ is (1)

Solution: The volume is given by the formula $|(\vec{u} \times \vec{v}) \cdot \vec{w}|$. First, we compute

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & 3 & 2 \end{vmatrix} = (2, -2, 2)$$

Then we find

$$(2, -2, 2) \cdot (1, 1, 3) = 6, \text{ and } |6| = 6.$$

`cross (X) the correct answer:`

- A 8
- B 3
- C 6 **- Correct**
- D 9
- E 1.5
- F 36

2. The distance from the point $P = (4, 0, 0)$ to the plane $2x - y + 2z = -3$ is: (1)

Solution: Choose a point $Q = (0, 3, 0)$ which belongs to the plane. Then $P - Q = (4, -3, 0)$. The normal vector to the plane is $\vec{n} = (2, -1, 2)$. So the distance is

$$||proj_{\vec{n}}(P - Q)|| = \frac{|\vec{n} \cdot (P - Q)|}{||\vec{n}||} = \frac{11}{3}$$

cross (X) the correct answer:

- A 9
- B 11
- C $\frac{11}{9}$
- D $\frac{11}{3}$ **- Correct**
- E $\sqrt{\frac{11}{3}}$
- F $\sqrt{11}$

3. Mark whether each of the following statements is TRUE or FALSE in the respective box. (each correct answer is 1/4pt)

- A homogeneous system of linear equations can have infinitely many solutions.

  ANSWER: ☐ TRUE

- It is possible that a system of linear equations with coefficients in $\mathbb{R}$ has exactly 1 solution.

  ANSWER: ☐ TRUE

- A homogeneous system of linear equations can be inconsistent.

  ANSWER: ☐ FALSE

- There exists a system of three equations such that its coefficient matrix has rank 7.

  ANSWER: ☐ FALSE

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4. If the coefficient matrix A in a homogeneous system in 22 variables of 15 equations is known to have rank 10, how many parameters are there in the general solution? (1)

Solution: Homogeneous system is consistent. So the number of parameters is $22 - 10 = 12$.

`cross (X) the correct answer`:

[A] 7

[B] 5

[C] 12 **- Correct**

[D] 15

[E] 22

[F] 10

5. Suppose $e, f \in \mathbb{R}$ and consider the linear system in x, y and z:

$$\begin{cases} 2x - 2y + 2ez & = 3f \\ x + z & = -1 \\ 3x + y + 2z & = -1 \end{cases}$$

5(a) If $(A \mid b)$ is the augmented matrix of the system above, find the rank of A and the rank of $(A \mid b)$ for **all** values of e and f. (2)

(you must justify all your answers: correct answer without justification is 1pt only)

Solution: Performing Gauss elimination we obtain

$$(A \mid b) = \left(\begin{array}{ccc|c} 2 & -2 & 2e & 3f \\ 1 & 0 & 1 & -1 \\ 3 & 1 & 2 & -1 \end{array}\right) \xrightarrow{\text{permute rows}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 3 & 1 & 2 & -1 \\ 2 & -2 & 2e & 3f \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 2e-2 & 3f+2 \end{array}\right)$$

$$\xrightarrow{\text{last row}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2e-4 & 3f+6 \end{array}\right)$$

Hence,

$$rank\, A = \begin{cases} 2 & if\ e = 2 \\ 3 & if\ e \neq 2 \end{cases}$$

$$rank\,(A \mid b) = \begin{cases} 2 & if\ e = 2\ and\ f = -2 \\ 3 & otherwise \end{cases}$$

5(b) Using part (a), find all values of e and f so that this system has

(i) a unique solution (1)

Solution: it has a unique solution $\iff rank\, A = rank\,(A \mid b) = 3 \iff e \neq 2$.

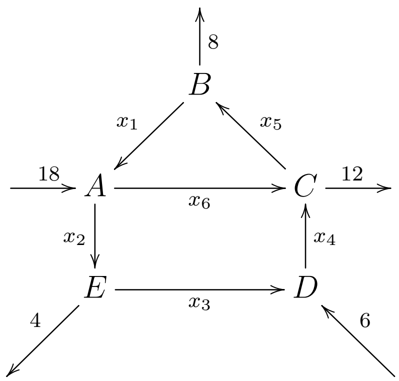
(ii) infinitely many solutions (1)

Solution: it has infinitely many solutions $\iff rank\, A = rank\,(A \mid b) < 3 \iff e = 2$ and $f = -2$.

(iii) no solutions (1)

Solution: it has no solutions $\iff rank\, A < rank\,(A \mid b) \iff e = 2$ and $f \neq -2$.

6. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the one-way streets, and the numbers refer to the exact number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passes along the indicated streets during the same period.



6(a) Write down a system of linear equations which describes the traffic flow **together with all the constrains on the variables** x_i, $i = 1, \ldots, 6$. (1)

(Do not perform any operations on your equations: this is done for you in (b). Do not simply copy out the equations implicit in (b). You will not get any marks if you do this)

Solution:

Intersection	Flow in = Flow out
A	$18 + x_1 = x_2 + x_6$
B	$x_5 = x_1 + 8$
C	$x_4 + x_6 = x_5 + 12$
D	$6 + x_3 = x_4$
E	$x_2 = x_3 + 4$

Oneway streets give constrains $x_i \geq 0$, $i = 1, \ldots, 6$. Since each x_i is a number of cars, x_i has to be an integer.

6(b) The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & -8 \\ 0 & 1 & 0 & 0 & -1 & 1 & 10 \\ 0 & 0 & 1 & 0 & -1 & 1 & 6 \\ 0 & 0 & 0 & 1 & -1 & 1 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Give the general solution of this system (1)

(Ignore the constrains from (a) at this point)

Solution: Set $x_5 = s$ and $x_6 = t$ to be parameters. Then

$$\begin{aligned} x_1 &= -8 + s \\ x_2 &= 10 + s - t \\ x_3 &= 6 + s - t \\ x_4 &= 12 + s - t \\ x_5 &= s \\ x_6 &= t \end{aligned}$$

6(c) If the road ED was closed in the middle due to roadwork, find the minimum flow along the road AC **using your results from (b)** (2)

(you must justify all your answers: correct answer without justification is 1pt only)

Solution: ED is closed $\iff x_3 = 0 \iff s - t = -6$. Then assuming the constrains $x_i \geq 0$ we obtain the system of inequalities

$$\begin{cases} x_1 & = -8 + s \geq 0 \iff s \geq 8 \\ x_2 & = 10 - 6 = 4 \\ x_3 & = 0 \\ x_4 & = 12 - 6 = 6 \\ x_5 & = s \geq 0 \\ x_6 & = s - 6 \geq 0 \iff s \geq -6 \end{cases}$$

The flow along AC is $x_6 = s + 6$ so that $x_6 \geq 14$. The minimum flow along AC is 14.

7. Mark whether each of the following statements is TRUE or FALSE in the respective box. (each correct answer is 1/4pt)

- For any two 3×3 matrices A and B, we have $(A-B)^2 = A^2 - 2AB + B^2$.

  ANSWER: ☐ FALSE

- There exist two non-zero matrices A and B such that $A \cdot B$ is the zero-matrix.

  ANSWER: ☐ TRUE

- Multiplying a 4×2-matrix A by a 2×4-matrix B one gets a 4×4-matrix AB.

  ANSWER: ☐ TRUE

- For any two matrices A and B we have $AB = BA$.

  ANSWER: ☐ FALSE

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8. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, and B is a 3×5 matrix,

then the second row of the matrix $A \cdot B$ is (1)

`cross (X) the correct answer`:

[A] the same as the first row of B

[B] the sum of the first and the second rows of B

[C] the sum of the first, the second and the third rows of B

[D] the sum of the first and the third rows of B - **Correct**

[E] the same as the second row of B

[F] the sum of the second and the third rows of B

Solution: Write $B = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$ in block form, where r_i is the i-th row of B. Then

$$A \cdot B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_1 + r_3 \\ r_1 + r_2 + r_3 \end{pmatrix}$$

The last page (use it for computations)