Solutions Midterm 2

1. a) $A-B=\{3,5,7\}$

b) $A\cup B=\{1,2,3,4,5,7,8,9\}$

$\overline{A\cup B}=\{6,10\}$

c) $A\oplus B=(A-B)\cup(B-A)=\{3,5,7\}\cup\{2,4,8\}$

$=\{2,3,4,5,7,8\}$

d) $(A\cup B)-(A\cap B)=\{1,2,3,4,5,7,8,9\}-\{1,9\}$

$=\{2,3,4,5,7,8\}$

e) $|A\cap B|=2$

f) $|P(A)|=2^{|A|}=2^5=32$

2. a) $(A\cup B)\subseteq(A\cup B\cup C)$

P: $x\in A\cup B \Rightarrow x\in A$ or $x\in B$ (or $x\in C$)

$\Rightarrow x\in A$ or $x\in B$

$\Rightarrow x\in A\cup B\cup C$

$\therefore A\cup B\subseteq A\cup B\cup C$

b) $A\cap B\cap C\subseteq A\cap B$

If $x\in A\cap B\cap C \Rightarrow x\in A$ and $x\in B$ and $x\in C$

$\Rightarrow x\in A$ and $x\in B \Rightarrow x\in A\cap B$

$\therefore A\cap B\cap C\subseteq A\cap B$

2c) $A \subseteq B \Rightarrow A \cup B = B$

$\subseteq$: If $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in B$ or $x \in B$ (as $A \subseteq B$)

$\Rightarrow x \in B$

$\therefore A \cup B \subseteq B$

$\supseteq$: If $x \in B \Rightarrow x \in B$ or $x \in A$

$\Rightarrow x \in A \cup B$

$\therefore B \subseteq A \cup B$

$\therefore A \subseteq B \Rightarrow A \cup B = B$

3: $\overline{(A - B)} \cap C = (C \cap A) \cup (C \cap B)$

$\overline{(A - B)} \cap C \iff \overline{(\overline{A} \cap B)} \cap C$ (1)

$\iff (\overline{\overline{A}} \cup \overline{\overline{B}}) \cap C$ (18)

$\iff (A \cup \overline{\overline{B}}) \cap C$ (9)

$\iff (A \cup B) \cap C$ (9)

$\iff C \cap (A \cup B)$ (12)

$\iff (C \cap A) \cup (C \cap B)$ (15)

4a) If $f(x) = f(y)$

$\Rightarrow x^3+1 = y^3+1 \Rightarrow x^3 = y^3$

$\Rightarrow \sqrt[3]{x^3} = \sqrt[3]{y^3}$

$\Rightarrow x = y$

$\therefore$ f is 1-to-1

For $y \in \mathbb{R}$, suppose $f(x) = x^3+1 = y$

$\Rightarrow x^3 = y-1 \Rightarrow x = (y-1)^{1/3}$

$\Rightarrow x = \sqrt[3]{y-1}$

<u>We check:</u> For $y \in \mathbb{R}$,

$f(\sqrt[3]{y-1}) = (\sqrt[3]{y-1})^3+1 = (y-1)+1 = y$

$\therefore$ f is onto

f is bijective since 1-to-1 and onto.

b) f^{-1} exists since f is bijective

We have $f \circ f^{-1} = i_{\mathbb{R}}$

For $x \in \mathbb{R}$, $(f \circ f^{-1})(x) = i_{\mathbb{R}}(x) = x$

$\Rightarrow f(f^{-1}(x)) = x$

$\Rightarrow (f^{-1}(x))^3 + 1 = x$

$\Rightarrow (f^{-1}(x))^3 = x-1 \Rightarrow f^{-1}(x) = \sqrt[3]{x-1}$

5. a) $(f \circ g)(x) = f(g(x))$

$\underline{f \circ g}$: $(f \circ g)(1) = f(g(1)) = f(1) = 2$

$(f \circ g)(2) = f(g(2)) = f(4) = 3$

$(f \circ g)(3) = f(g(3)) = f(2) = 1$

$(f \circ g)(4) = f(g(4)) = f(3) = 4$

b) $(h \circ f)(1) = h(f(1)) = h(2) = 3$

$(h \circ f)(2) = h(f(2)) = h(1) = 4$

$(h \circ f)(3) = h(f(3)) = h(4) = 1$

$(h \circ f)(4) = h(f(4)) = h(3) = 2$

$\underline{6.}$ $\underline{R}$: For $(a,b) \in \mathbb{N}^2$

$a+b = b+a \Rightarrow ((a,b),(a,b)) \in R$

Hence, yes.

$\underline{S}$: If $((a,b),(c,d)) \in R \Rightarrow a+d = b+c$

$\Rightarrow b+c = a+d$

$\Rightarrow c+b = d+a$

$\Rightarrow ((c,d),(a,b)) \in R$

$\underline{T}$: If $((a,b),(c,d)) \in R$

and $((c,d),(e,f)) \in R$

$\Rightarrow a+d = b+c \quad \Rightarrow a = b+c-d$

$c+f = d+e \quad f = d+e-c$

$\Rightarrow a+f = (b+c-d)+(d+e-c) = b+e$

$\Rightarrow ((a,b),(e,f)) \in R \quad \therefore$ Yes.

7. $\{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a \geq b^2\}$

R: For $a \in \mathbb{Z}$, $a \geq a^2$ is not always true.

For example, $2 \not\geq 2^2 = 4 \Rightarrow (2,2) \notin R$

Hence, no.

S: No. $(4,2) \in R$ but $(2,4) \notin R$ as $2 \not\geq 4^2 = 16$

A: If $(a,b) \in R$ and $(b,a) \in R$

$\Rightarrow a \geq b^2$ and $b \geq a^2 \Rightarrow a \geq a^4$

$\Rightarrow a = 0$ or $a = 1$

(since they are the only values where this is true.)

If $a = 1$, then $1 \geq b^2$ and $b \geq 1 \Rightarrow b = 1$

So $(a,b) = (1,1)$

If $a = 0$, then $0 \geq b^2 \Rightarrow b = 0$ since $b \in \mathbb{Z}$

Hence, $(a,b) = (0,0)$.

The only time we have $(a,b) = (b,a)$ implies that $(a,b) = (0,0)$ or $(a,b) = (1,1)$

In any cases, $a = b \Rightarrow$ Yes.

T: Yes. If $(a,b) \in R$ and $(b,c) \in R$

$\Rightarrow a \geq b^2$ and $b \geq c^2$

$\Rightarrow a \geq b^2 \geq (c^2)^2 = c^4$ But for $c \in \mathbb{Z}$, $c^4 \geq c^2$ is always true.

Hence, $a \geq b^2 \geq c^4 \geq c^2$

$\Rightarrow (a,c) \in R$.

8\. a) $[10101]_R = \{10001, 10011, 10101, 11001, 10111, 11101, 11011, 11111\}$

b) $[01011]_R = \{00001, 00011, 00101, 01001, 01101, 01011, 00111, 01111\}$

9\. $[(3,4)]_R = \{(x,y) \in A \mid ((3,4),(x,y)) \in R\}$

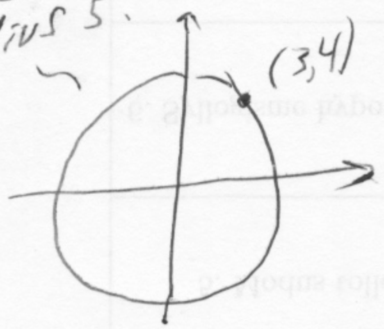
$= \{(x,y) \in A \mid 3^2+4^2 = x^2+y^2\}$

$= \{(x,y) \in A \mid x^2+y^2 = 25\}$

the equation of a circle of radius 5 centered at $(0,0)$

= all the points on the circle of radius centered at $(0,0)$

circle of radius 5.



10\. For $(a,b) \in A$, $[(a,b)]_R = \{(x,y) \in A \mid x \equiv a(2), y \equiv b(5)\}$

For $a \in \mathbb{Z}$, we have: $a \equiv 0(2)$ or $a \equiv 1(2)$

For $b \in \mathbb{Z}$, we have: $b \equiv 0(5)$, $b \equiv 1(5)$, $b \equiv 2(5)$, $b \equiv 3(5)$ or $b \equiv 4(5)$.

The induced partition is thus:

$\{[0]_2 \times [0]_5, [0]_2 \times [1]_5, [0]_2 \times [2]_5, [0]_2 \times [3]_5, [0]_2 \times [4]_5,$
$[1]_2 \times [0]_5, [1]_2 \times [1]_5, [1]_2 \times [2]_5, [1]_2 \times [3]_5, [1]_2 \times [4]_5\}$