2020 Sample Midterm 1, pieced together from questions from previous years. (The difference is: we'll have fewer questions on high school review directly and no questions on infinite limits.)

$f(x)$ is continuous at a if $\lim_{x\to a} f(x)$ exists and $f(a) = \lim_{x\to a} f(x)$.

5. (1 point) Does there exist a choice of a which makes the following function continuous at $x = 1$? If yes, which one? If not, why not?

$$f(x) = \begin{cases} \ln(2ax) & \text{if } x > 1 \\ \cos(\pi x) & \text{if } x \leq 1 \end{cases}$$

(and 2 points for work)

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \ln(2ax) = \ln(2a)$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \cos(\pi x) = \cos(\pi) = -1$

we require $\lim_{x\to 1^-} = \lim_{x\to 1^+}$

$\to \ln(2a) = -1$

$\to 2a = e^{-1}$

$\to a = \dfrac{e^{-1}}{2} = \dfrac{1}{2e}$

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7. (2 points) Evaluate the limit $\lim_{x\to 4^-} \dfrac{x^2 - x - 12}{|x-4|}$, if it exists. **You must use algebraic methods and justify your answer mathematically to earn credit for this question.**

$= \lim_{x\to 4^-} \dfrac{(x-4)(x+3)}{-(x-4)} = \lim_{x\to 4^-} -(x+3) = -7$

$|x-4| = \begin{cases} (x-4) & x \geq 4 \\ -(x-4) & x < 4 \end{cases}$

8. (1 point) A study started tracking a herd of elk in northern Alberta. In January 1996, there were 1000 elk in the herd. Over the course of this study, scientists observed that the elk had a natural growth rate of 5% each year. Furthermore, in December each year, an average of 50 elk were killed by hunters. If h_t represents the herd's population (in January), t years after 1996, which of the following could be a DTDS modeling the dynamics of this herd?

A. $h_{t+1} = 5h_t + 950$

B. $h_{t+1} = 50h_t - 1000$

C. $h_{t+1} = 1.05h_t - 50$

D. $h_{t+1} = 1.05h_t + 950$

E. $h_{t+1} = 0.05h_t - 50$

F. $h_{t+1} = \dfrac{50}{1000}h_t + 0.05$

Your answer: C

Initial: $h_0 = 1000$

Growth rate of 5%, so $r = 1.05$

each year we lose 50 elk

So $h_{t+1} = 1.05h_t - 50$

10. (1+1+2+1=5 points)

A population of snails grows according to a Ricker model, so that the DTDS governing the growth of this population's density is given by

$$x_{t+1} = 2x_t e^{1-0.4x_t}$$

where t is in months and x_t denotes the number of snails per m^2 at time t.

(a) Give the updating function of this DTDS. $f(x) = \boxed{2xe^{1-0.4x}}$

(b) Find all equilibria of this DTDS. Show your work.

Set $x^* = 2x^* e^{1-0.4x^*} \rightarrow x^* - 2x^* e^{1-0.4x^*} = 0$

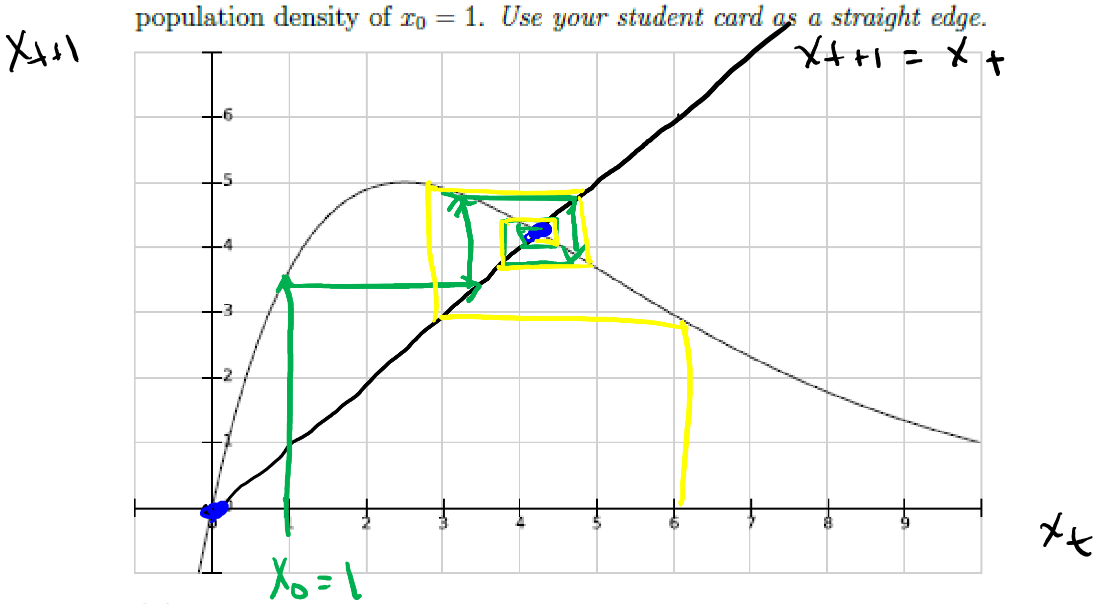
$\rightarrow x^*(1 - 2e^{1-0.4x^*}) = 0$

$x^* = 0, \quad e^{1-0.4x^*} = 1/2 \rightarrow 1 - 0.4x^* = \ln(1/2)$

$x^* = \dfrac{1+\ln(2)}{0.4}$

$x^* = \boxed{0, \quad \dfrac{1+\ln(2)}{0.4}}$

(c) The graph of $y = f(x)$ is drawn below. **Label the axes**, identify the **fixed points** of the DTDS on the graph, then **draw a cobweb** for at least three iterations, starting with an initial population density of $x_0 = 1$. *Use your student card as a straight edge.*



(d) What do you conclude will happen to this population of snails in the long term? (Your answer should be a complete sentence and should refer to the concepts of stability and fixed points as they pertain to this population of snails and your cobweb.)

The fixed point $x^* = \dfrac{1+\ln(2)}{4}$ is stable since the cobweb goes towards it. Therefore the population will go to this value in the long term.

for $x_{t+1} = r x_t + c$, $x_t = r^t(x_0 - x^*) + x^*$

11. (1+2+1=4 points) A patient is given a daily dose of the drug ToxicPlacebo, which is absorbed into the patient's system at a constant rate. A DTDS modeling the concentration of drug x_t, in mg/L in the patient's bloodstream each day just after receiving the dose is given by

$$x_{t+1} = 0.9x_t + 0.5,$$

where t is measured in whole days since the treatment began.

(a) Find the fixed point of this DTDS, and give the general solution to this DTDS if the initial value is $x_0 = 1$ mg/L.

$x^* = 0.9x^* + 0.5$

$x_t = 0.9^t(1-5) + 5$
$= -4(0.9)^t + 5$

Fixed point: $x^* = \boxed{5}$ $\rightarrow 0.1x^* = 0.5$ $\rightarrow x^* = 5$

General solution: $\boxed{x_t} = \boxed{5 - 4(0.9)^t}$

find t

(b) If the initial value is $x_0 = 1$ mg/L, find the number of whole days it takes for the concentration of ToxicPlacebo in the patient's bloodstream to reach at least 4.5 mg/L. $\rightarrow x_t = 4.5$

$5 - 4(0.9)^t = 4.5$

$\rightarrow 0.5 = 4(0.9)^t$

$\rightarrow 0.125 = (0.9)^t$

$\rightarrow \ln(0.125) = t\ln(0.9)$

$\rightarrow t = \dfrac{\ln(0.125)}{\ln(0.9)} \approx 19.74$

Minimum number of whole days required: $\boxed{20}$

(c) Surprisingly, testing over the year reveals that the concentration of ToxicPlacebo in the patient's bloodstream has stabilized to $x^* = 6$ mg/L. An investigation reveals that the doctor's bad handwriting was to blame, and the daily administered dose c in reality was not actually 0.5 mg/L. What was it?

$x^* = 0.9x^* + c$

$\rightarrow 6 = (0.9)(6) + c$

$\rightarrow 6 - 5.4 = c$

$\rightarrow c = 0.6$ mg/L

So the proper DTDS is

$x_{t+1} = 0.9x_t + 0.6$

9. (4 points) Decide if the following limits exist. For each one, if it exists, evaluate the limit exactly using algebraic methods, showing all steps. If it does not exist, justify your answer clearly using mathematical reasoning.

(b) (2 points) $\lim_{x \to 4} \frac{3 - \sqrt{25 - x^2}}{x - 4}$ $\frac{(3+\sqrt{25-x^2})}{(3+\sqrt{25-x^2})}$

$= \lim_{x \to 4} \frac{9 - (25 - x^2)}{(x-4)(3+\sqrt{25-x^2})}$

$= \lim_{x \to 4} \frac{-16 + x^2}{(x-4)(3+\sqrt{25-x^2})}$

$= \lim_{x \to 4} \frac{\cancel{(x-4)}(x+4)}{\cancel{(x-4)}(3+\sqrt{25-x^2})}$

$= \lim_{x \to 4} \frac{x+4}{3+\sqrt{25-x^2}}$

$= \frac{8}{3+\sqrt{25-16}}$

$= \frac{8}{3+\sqrt{9}}$

$= 8/6$

$\therefore \lim_{x \to 4} \frac{3 - \sqrt{25-x^2}}{x-4} = 8/6$