3. Discrete-time Dynamical Systems (DTDS)

A **discrete-time dynamical system (DTDS)** is

- a quantity x whose change is tracked at time t
  - → write x_t to represent "x at time t"
- a suitable time step
  - → difference between time t and time $t+1$
- an updating function $f(x)$
  - → describes the change in x after one time step, i.e., $x_{t+1} = f(x_t)$
- for short, the DTDS is summarized as $\boxed{x_{t+1} = f(x_t)}$
- the updating function $f(x)$ of a DTDS can be
  - simple (ie linear)
  - complicated (ie nonlinear)
- $f(x)$ captures the change that occurs to the quantity from one time step to the next.

FIXED POINTS (EQUILIBRIA) OF A DTDS

A point x^* of a DTDS $\boxed{x_{t+1} = f(x_t)}$ is called an **equilibrium** or a **fixed point** if

$$f(x^*) = x^*.$$

- If the quantity $x = x^*$, then the DTDS leaves x^* unchanged at each time step.
- If $x_t = x^*$, then $x_{t+1} = f(x_t) = f(x^*) = x^* = x_t \quad \therefore \ x_{t+1} = x_t$

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* These notes are solely for the personal use of students registered in MAT1330.

Example 3.1. A bacteria population doubles every 24 hours. Give the DTDS that models this population. What (if any) are the fixed points?

- quantity x_t = # bacteria on day t
- time step 24 hours between time t and time $t+1$
- updating function $f(x)$ $f(x) = 2x$ (f is a linear function)
- DTDS

$$x_{t+1} = f(x_t) = 2x_t$$

$x_{tomorrow} = 2(x_{today})$

bacteria tomorrow = double the # bacteria today

- fixed points? solve $x = f(x)$: $x = 2x \Rightarrow 0 = x$
  ∴ there is one fixed point: $x^* = 0$

  (0 bacteria today ⇒ 0 bacteria tomorrow)

Example 3.2. Bamboo is one of the fastest growing plants on Earth with certain species having a growth rate of 3 cm / hour. Give the DTDS that models bamboo height. What (if any) are the fixed points?

- quantity h_t = height (in cm) of bamboo at hour t
- time step 1 hour between time t and time $t+1$
- updating function $f(x)$ $f(x) = x + 3$
- DTDS

$$h_{t+1} = f(h_t) = h_t + 3$$

$h_{one\ hour\ from\ now} = h_{now} + 3$

height of bamboo one hour from now = height of bamboo now + 3cm.

- fixed points? solve $x = f(x)$: $x = x + 3 \Rightarrow 0 = 3$ ↯
  the equation $x = f(x)$ has no solutions
  ∴ this DTDS has no fixed points

Example 3.3. A patient is given an initial dose ("loading dose") of pain medication at the start of her treatment. Afterward, one dose of the medication is administered at the end of each day. It is known that the patient's body will absorb 50% of the given medication by the end of the day. Give the DTDS that models the daily change in the amount (in # doses) of the medication that is present in her body (not yet absorbed/used up) at the end of each day (just after the daily dose is given).

- quantity M_t = amount (# doses) of med. in patient's body just after daily dose is given
- time step 1 day between time t and time $t+1$
- updating function $f(x)$ $\quad f(x) = 0.5x + 1$
- DTDS $\boxed{M_{t+1} = f(M_t) = 0.5M_t + 1}$

amount in body just after daily dose is given = half of what was in body one day ago is absorbed, leaving half M_t and 1 new dose is given.

- fixed points? solve $x = f(x)$: $x = 0.5x + 1 \Rightarrow x = 2$

  ∴ this DTDS has one fixed point: $M^* = 2$ check: $f(2) = 0.5(2) + 1 = 2$ ✓

Calculate the amount in her body at the end of each of the first 4 days of treatment assuming her loading dose was

a. a single dose

$M_0 = 1$

$M_1 = 0.5M_0 + 1 = 0.5(1) + 1 = 1.5$

$M_2 = 0.5M_1 + 1 = 0.5(1.5) + 1 = 1.75$

$M_3 = 0.5M_2 + 1 = 0.5(1.75) + 1 = 1.875$

$M_4 = 0.5M_3 + 1 = 1.9375$

b. a triple dose

$M_0 = 3$

$M_1 = 0.5M_0 + 1 = 0.5(3) + 1 = 2.5$

$M_2 = 0.5M_1 + 1 = 0.5(2.5) + 1 = 2.25$

$M_3 = 0.5M_2 + 1 = 0.5(2.25) + 1 = 2.125$

$M_4 = 0.5M_3 + 1 = 2.0625$

c. a double dose

$M_0 = 2$

$M_1 = 0.5M_0 + 1 = 0.5(2) + 1 = 2$

$M_2 = 0.5M_1 + 1 = 0.5(2) + 1 = 2$

$M_3 = 2$

$M_4 = 2$

⋮

When $M_t = 2$, it follows that $M_{t+1} = M_t = 2$

Let $x_{t+1} = f(x_t)$ be a DTDS. A **solution** to the DTDS $x_{t+1} = f(x_t)$ with **initial condition** x_0 is

the sequence $\{x_0, x_1, x_2, x_3, \dots\}$

That is, the solution to a DTDS with I.C. x_0 is the entire future of the system, starting from x_0.

Example 3.4. Bamboo DTDS. $h_{t+1} = f(h_t) = h_t + 3$ $\quad (f(x) = x+3)$

h_0 (given)

$h_1 = f(h_0) = h_0 + 3$

$h_2 = f(h_1) = h_1 + 3 = (h_0+3)+3 = h_0+6$

$h_3 = f(h_2) = h_2 + 3 = (h_0+6)+3 = h_0+9$

$h_4 = f(h_3) = h_3 + 3 = (h_0+9)+3 = h_0+12$

$\vdots$

pattern: $h_t = h_0 + 3t$

solution

$\{h_0, h_0+3, h_0+6, h_0+9, \dots\}$

general solution

$h_t = h_0 + 3t$

(with each time step initial height increases by 3 cm)

Example 3.5. Bacteria DTDS. $x_{t+1} = f(x_t) = 2x_t$ $\quad f(x) = 2x$

x_0 (given)

$x_1 = f(x_0) = 2x_0$

$x_2 = f(x_1) = 2x_1 = 2(2x_0) = 4x_0$

$x_3 = f(x_2) = 2x_2 = 2(4x_0) = 8x_0$

$x_4 = f(x_3) = 2x_3 = 2(8x_0) = 16x_0$

$\vdots$

general pattern: $x_t = 2^t x_0$

Solution

$\{x_0, 2x_0, 4x_0, 8x_0, 16x_0, \dots\}$

Ex general solution with $x_0 = 50$

$x_t = 2^t \cdot 50$

Ex $x_6 = 2^6 \cdot 50 = (64)(50) = 3200$

Ex How many days are needed for # bacteria to be at least 10 000 (with $x_0 = 50$)?

Solve for t in $10\,000 \leq x_t$

$10\,000 \leq 2^t \cdot 50$

$\Rightarrow 200 \leq 2^t$

$\Rightarrow \ln 200 \leq t \ln 2$

$\Rightarrow t \geq \frac{\ln 200}{\ln 2} \approx 7.64$

$\therefore$ In 8 days, there will be $\geq 10\,000$ bacteria.

Example 3.6. Medication DTDS. $M_{t+1} = f(M_t) = 0.5M_t + 1$ $\qquad f(x) = 0.5x + 1$

M_0 given ("loading dose")

$M_1 = f(M_0) = 0.5M_0 + 1$

$$M_2 = f(M_1) = 0.5M_1 + 1 = 0.5(0.5M_0 + 1) + 1 = (0.5)^2 + 0.5 + 1$$

$$M_3 = f(M_2) = 0.5M_2 + 1 = 0.5\left((0.5)^2M_0 + 0.5 + 1\right) + 1 = (0.5)^3M_0 + 0.5^2 + 0.5 + 1$$

$\vdots$

particular solution with $M_0 = 1$

$\{1, 1.5, 1.75, 1.875, 1.9375, \ldots\}$

particular solution with $M_0 = 3$

$\{3, 2.5, 2.25, 2.125, 2.0625, \ldots\}$

particular solution with $M_0 = 2$

$\{2, 2, 2, 2, 2, \ldots\}$

General Solution of a DTDS with a Linear Updating Function

$$f(x) = mx + b \qquad (m > 0)$$

Special case: $m = 1$ $\quad f(x) = x + b$

Solution: $\{x_0, x_0 + b, x_0 + 2b, x_0 + 3b, \ldots\}$

More interesting case: $m \neq 1$ $\quad f(x) = mx + b$

x_0 (given I.C.)

$x_1 = mx_0 + b$

$$x_2 = mx_1 + b = m(\underbrace{mx_0 + b}_{x_1}) + b = m^2x_0 + b(m+1)$$

$$x_3 = mx_2 + b = m(\underbrace{m^2x_0 + b(m+1)}_{x_2}) + b = m^3x_0 + b(m^2 + m + 1)$$

$$x_4 = m(\underbrace{m^3x_0 + b(m^2 + m + 1)}_{x_3}) + b = m^4x_0 + b(m^3 + m^2 + m + 1)$$

$\vdots$

the general solution is

$$x_t = m^tx_0 + b(m^{t-1} + m^{t-2} + \ldots + m^2 + m + 1)$$

↑ a bit tedious to compute

Ex $x_{100} = ?$ ☹

Sum of a Geometric Series.

general solution to DTDS with $f(x)=mx+b$, $m\neq 1$ looks like

$$x_t = m^t x_0 + b(\underbrace{m^{t-1}+m^{t-2}+\dots+m+1}_{\text{this term is a geometric series}})$$

Let $S = m^{t-1}+m^{t-2}+\dots+m+1$

Then $mS = m(m^{t-1}+m^{t-2}+\dots+m+1)$

$= m^t + \underbrace{m^{t-1}+\dots+m^2+m}$

$= m^t + S - 1$

Thus, $mS = m^t + S - 1$ → Now solve for S:

$\Rightarrow mS - S = m^t - 1$

$\Rightarrow S(m-1) = m^t - 1$

$\Rightarrow \boxed{S = \dfrac{m^t-1}{m-1}}$ ← recall $m\neq 1$

Solution to DTDS with Linear Updating Function.

$x_t = m^t x_0 + bS$

$= m^t x_0 + b\left(\dfrac{m^t-1}{m-1}\right)$

$= m^t x_0 - b\left(\dfrac{m^t-1}{1-m}\right)$

$= m^t x_0 - \dfrac{b\cdot m^t}{1-m} + \dfrac{b}{1-m}$

General Solution to DTDS with $f(x)=mx+b$, $m\neq 1$ is now simplified to $\boxed{x_t = m^t\left(x_0 - \dfrac{b}{1-m}\right) + \dfrac{b}{1-m}}$

Let's let $x^* = \dfrac{b}{1-m}$

$\Rightarrow$ General Solution is $\boxed{x_t = m^t(x_0 - x^*) + x^*}$

Example 3.7. Medication DTDS Solution. $f(x) = 0.5x + 1$ (so $m=0.5$, $b=1$) and $x^* = \dfrac{b}{1-m} = 2$

∴ the general solution to this DTDS is

$$M_t = 0.5^t(x_0 - 2) + 2$$

Ex with $x_0 = 1$, we can find x_7 directly:
$x_7 = 0.5^7(1-2)+2 \approx 1.992\dots$

DTDS with linear updating function: $f(x) = mx+b$. $\quad x_{t+1} = mx_t + b$

$m=1$: $f(x) = x+b \quad x_{t+1} = x_t + b \Rightarrow$ general solution is $x_t = x_0 + tb$

$m\neq 1$, $b=0$: $f(x) = mx \quad x_{t+1} = mx_t \Rightarrow$ general solution is $x_t = m^t x_0$

$m\neq 1$, $b\neq 0$: $f(x) = mx+b \quad x_{t+1} = mx_t + b$

$\Rightarrow$ general solution is $x_t = m^t(x_0 - x^*) + x^*$ where $x^* = \dfrac{b}{1-m}$

Example 3.8. Suppose we have a DTDS given by $x_{t+1} = f(x_t)$. What would the composition $(f \circ f)(x)$ mean with respect to this DTDS?

$x_1 = f(x_0)$
$x_2 = f(x_1)$
$\vdots$
$x_{t+1} = f(x_t)$
$x_{t+2} = f(x_{t+1})$

$(f \circ f)(x_t) = f(f(x_t))$
$= f(x_{t+1})$
$= x_{t+2}$

The composition gives us a 2-time-steps DTDS

$$x_{t+2} = (f \circ f)(x_t)$$

Example 3.9. Suppose we have a DTDS given by $x_{t+1} = f(x_t)$. If f has an inverse, what does $f^{-1}(x)$ mean with respect to this DTDS?

$x_1 = f(x_0) \iff f^{-1}(x_1) = x_0$
$x_2 = f(x_1) \iff f^{-1}(x_2) = x_1$
$\vdots$
$x_{t+1} = f(x_t) \iff f^{-1}(x_{t+1}) = x_t$
$x_{t+2} = f(x_{t+1}) \iff f^{-1}(x_{t+2}) = x_{t+1}$

Inverse $y = f(x) \iff x = f^{-1}(y)$
thus $x_{t+1} = f(x_t) \iff x_t = f^{-1}(x_{t+1})$

The inverse gives us a back-one-time-step DTDS $\boxed{x_t = f^{-1}(x_{t+1})}$

Exercise 3.10. For the DTDS examples given (bacteria and bamboo), compute the composition $f \circ f$ and, if it exits, the inverse f^{-1} of their respective updating functions. Interpret these new functions in terms of the applications.

Bacteria
$f(x) = 2x$ $\boxed{x_{t+1} = f(x_t) = 2x}$

$(f \circ f)(x) = f(f(x)) = f(2x) = 2(2x) = 4x$

2-timesteps DTDS:

$$x_{t+2} = (f \circ f)(x_t) = 4x_t$$

bacteria 48 hours from now = 4 times the # of bacteria now

Bamboo
$f(x) = x+3$ $\boxed{h_{t+1} = f(h_t) = h_t + 3}$

to find f^{-1}: $h_{t+1} = h_t + 3 \iff h_t = h_{t+1} - 3$

back-1-time-step DTDS:

$$h_t = f^{-1}(h_{t+1}) = h_{t+1} - 3$$

height of bamboo 1 hour ago = height now −3cm

STUDY GUIDE

Important terms and concepts:

- ◇ **updating function** $x_{t+1} = f(x_t)$
- ◇ **equilibrium / fixed point of DTDS with updating function** $f(x)$:

  $\boxed{\text{a point } x^* \text{ such that } x^* = f(x^*)}$

- ◇ **solution of a DTDS with initial condition** x_0: $x_0, x_1, x_2, \ldots$
- ◇ **two-time-step updating function** $x_{t+2} = (f \circ f)(x_t)$
- ◇ **back-one-time-step updating function** $f^{-1}(x_{t+1}) = x_t$
- ◇ **general solution of a DTDS with a linear updating function**: $x_t = r^t(x_0 - x^*) + x^*$

Adler & Lovrić, 2nd ed.	§3.1, pg. 126 # 1–14, 17–31
Course Guide (on Brightspace)	§3.1 pg. 10 # 1–5
DGD Workbook (on Brightspace)	pg. 7–9 LEC 3