2. Review of High School

COMPOSITION

Let f and g be functions. If all numbers in the range of g are in the domain of f, then the **composition** $f \circ g$ is a function defined by $(f\circ g)(x) = f(g(x))$

Example 2.1. Find the composition $f \circ g$, where $f(x) = \dfrac{x-1}{x+1}$ and $g(x) = \dfrac{1}{\sqrt{x}}$

$$f\circ g(x) = f(g(x)) = f\left(\frac{1}{\sqrt{x}}\right) = \frac{\left(\frac{1}{\sqrt{x}}\right)-1}{\left(\frac{1}{\sqrt{x}}\right)+1} = \frac{\frac{1}{\sqrt{x}} - \frac{\sqrt{x}}{\sqrt{x}}}{\frac{1}{\sqrt{x}} + \frac{\sqrt{x}}{\sqrt{x}}} = \frac{\frac{1-\sqrt{x}}{\sqrt{x}}}{\frac{1+\sqrt{x}}{\sqrt{x}}} = \frac{(1-\sqrt{x})(\sqrt{x})}{(1+\sqrt{x})(\sqrt{x})} = \frac{1-\sqrt{x}}{1+\sqrt{x}}$$

cancels because $x \neq 0$

$\therefore f\circ g(x) = \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$

Bonus: Rationalize the denom. $\dfrac{1-\sqrt{x}}{1+\sqrt{x}} = \left(\dfrac{1-\sqrt{x}}{1+\sqrt{x}}\right)\left(\dfrac{1-\sqrt{x}}{1-\sqrt{x}}\right) = \dfrac{1-2\sqrt{x}+x}{1-x}$

What about $g\circ f$?

$$(g\circ f)(x) = g(f(x)) = g\left(\frac{x-1}{x+1}\right) = \frac{1}{\sqrt{\frac{x-1}{x+1}}}$$

Exercise 2.2. Use the table to evaluate each of the following expressions.

x	1	2	3	4	5	6
$f(x)$	3	1	4	2	2	5
$g(x)$	6	3	2	1	2	3

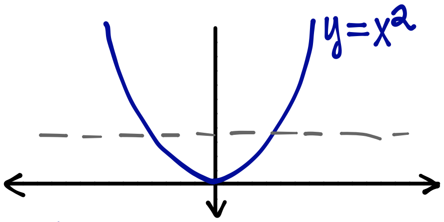
a. $f(g(1))$ $= f(6) = 5$ **b.** $(g\circ f)(3)$ $= g(4) = 1$ **c.** $(f\circ g)(6)$ $= f(3) = 4$

d. $(f\circ f)(1)$ $= f(3) = 4$ **e.** $g(g(1))$ $= g(6) = 3$ **f.** $(g\circ f)(1)$ $= g(3) = 2$

* These notes are solely for the personal use of students registered in MAT1330.

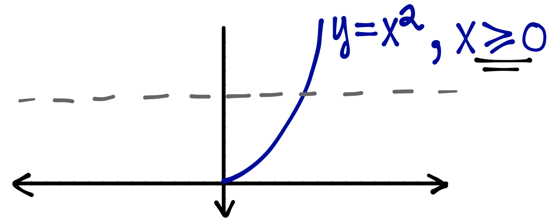
Horizontal Line Test: Let $y = f(x)$ be a function. If every horizontal line crosses the graph of f at most once, then $f(x)$ is a **one-to-one (injective)** function and f has an inverse.

Ex.



fails H.L.T. $\Rightarrow$ not one-to-one

Ex.



passes H.L.T. $\Rightarrow$ it's one-to-one!

Inverse: Let $y = f(x)$ be a function. If f passes the Horizontal Line Test, then the map f^{-1} defined by the rule

$$f^{-1}(y) = x \iff f(x) = y$$

is a function called the **inverse** of f. Informally, f^{-1} undoes f

Composition of a function with its inverse: $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$

Example 2.3. Find the inverse of $g(x) = \dfrac{2x-1}{3x+2}$ and verify that $(g \circ g^{-1})(x) = x = (g^{-1} \circ g)(x)$.

1. Write $y = g(x)$: $y = \dfrac{2x-1}{3x+2}$

2. Interchange x's and y's: $x = \dfrac{2y-1}{3y+2}$

3. Isolate 'new' y: $x(3y+2) = (2y-1)$

$\Rightarrow 3xy + 2x = 2y - 1$

$\Rightarrow 3xy - 2y = -2x - 1$

$\Rightarrow y(3x-2) = -2x - 1$

$\Rightarrow y = \dfrac{-2x-1}{3x-2}$

$\therefore g^{-1}(x) = \dfrac{-2x-1}{3x-2}$

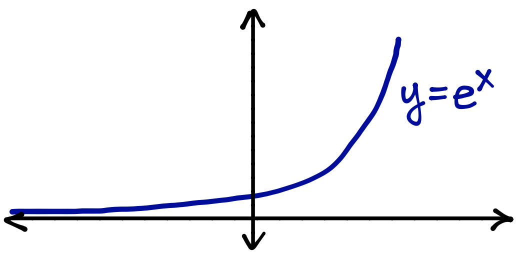
$$\begin{aligned}(g \circ g^{-1})(x) &= g(g^{-1}(x)) \\ &= g\left(\frac{-2x-1}{3x-2}\right) \\ &= \frac{2\left(\frac{-2x-1}{3x-2}\right) - 1}{3\left(\frac{-2x-1}{3x-2}\right) + 2} \\ &= \frac{\frac{-4x-2}{3x-2} - \frac{(3x-2)}{(3x-2)}}{\frac{-6x-3}{3x-2} + \frac{2(3x-2)}{3x-2}} \\ &= \frac{\frac{-7x}{3x-2}}{\frac{-7}{3x-2}} \\ &= \frac{-7x}{-7} \\ &= x \checkmark\end{aligned}$$

similarly, we can also verify that $(g^{-1} \circ g)(x) = x$.

Exponential Functions: $f(x) = a^x$ where $a > 0$ is a positive constant, $a \neq 1$

a is called the <u>base</u>

Natural Base: $f(x) = e^x$ $\quad e \approx 2.718281\ldots$



domain $(-\infty, \infty)$

range $(0, \infty)$

Laws of Exponents:

$$a^x a^y = a^{x+y} \qquad a^1 = a \qquad (ab)^x = a^x b^x$$

$$(a^x)^y = a^{xy} \qquad a^0 = 1$$

$$a^{-x} = \frac{1}{a^x} \qquad \frac{a^x}{a^y} = a^{x-y} \qquad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

Example 2.4. Solve for x in the equation $2^{x+3} = 16^{2x-1}$.

$$2^{x+3} = 16^{2x-1}$$
$$\Rightarrow 2^{x+3} = (2^4)^{2x-1}$$
$$\Rightarrow 2^{x+3} = 2^{8x-4}$$
$$\Rightarrow x+3 = 8x-4$$
$$\Rightarrow 7 = 7x$$
$$\Rightarrow x = 1$$

Check:

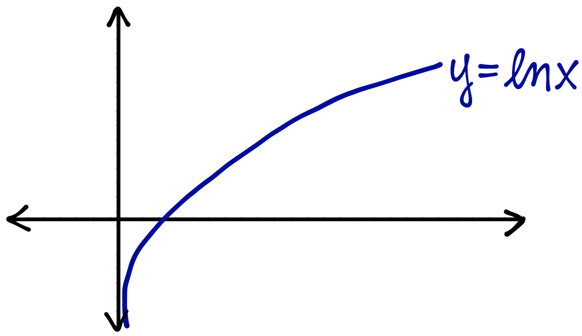
$LS = 2^{1+3} = 16 \qquad RS = 16^{2(1)-1} = 16$

$\therefore LS = RS$ ✓

Logarithmic Functions: $f(x) = \log_a(x)$ base $a > 0$, $a \neq 1$, domain: $(0, \infty)$

$$\log_a(x) = y \iff a^y = x \quad (x > 0)$$

Natural Logarithm: $f(x) = \ln(x)$ ← base e gets its own notation: $\log_e(x) = \ln x$



domain: $\{x \in \mathbb{R} : x > 0\}$

range: $\mathbb{R}$

Laws of Logs:

$\ln(xy) = \ln(x) + \ln(y)$

$\ln(x^p) = p\ln(x)$

$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$

$\ln(e) = 1$

$\ln(1) = 0$

$e^{\ln x} = x$

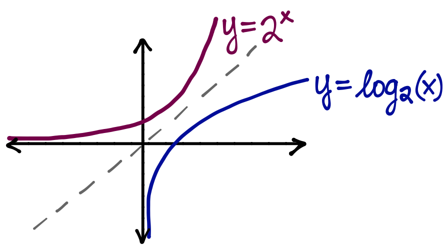
Change of base:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

Inverse Relationship between a^x and $\log_a(x)$:

$$\log_a(x) = y \iff a^y = x \quad (x > 0)$$

$\implies y = \log_a(x)$ is the inverse of $y = a^x$ and vice versa



composition of inverses

$\log_a(a^y) = y$

$\ln(e^y) = y$

$a^{\log_a(x)} = x$

$e^{\ln x} = x$

Example 2.5. Solve for x in the equation $\log(x+1) + \log(x+4) = 1$.

First, observe that $x > -1$ and $x > -4$ $\therefore$ $x > -1$

[in order for the equation to make sense, we must respect the domains of both $\log(x+1)$ and $\log(x+4)$]

$$\log(x+1) + \log(x+4) = 1$$
$$\Rightarrow \log\big((x+1)(x+4)\big) = 1$$
$$\Rightarrow \log(x^2+5x+4) = 1$$
$$\Rightarrow 10^{\log(x^2+5x+4)} = 10^1$$
$$\Rightarrow x^2+5x+4 = 10$$
$$\Rightarrow x^2+5x-6 = 0$$
$$\Rightarrow (x+6)(x-1) = 0$$
$$\Rightarrow x=-6 \text{ or } x=1$$

~~$x=-6$~~ reject because x has to be >-1

$\therefore$ the equation has one solution: $x=1$

Exercise 2.6. A bacterial population grows according to the model $b(t) = 1.8^t b_0$ where t represents time in hours, $b(t)$ represents the number of bacteria in the population at time t, and b_0 represents the initial population at time $t = 0$ (assume $b_0 > 0$).

How long will it take for the initial population to triple in size?

At what time t is $b(t) = 3b_0$?

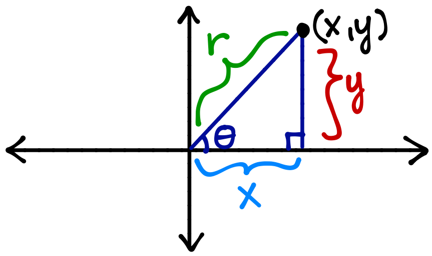
$$3b_0 = 1.8^t b_0$$
$$\Rightarrow 3 = 1.8^t \quad (\text{since } b_0 \neq 0)$$
$$\Rightarrow \ln(3) = \ln(1.8^t)$$
$$\Rightarrow \ln(3) = t \cdot \ln(1.8)$$
$$\Rightarrow t = \frac{\ln(3)}{\ln(1.8)} \approx 1.869\ldots$$

$\therefore$ it will take

$$t = \frac{\ln 3}{\ln 1.8} \approx 1.869 \text{ hours}$$

for the initial population to triple.

Trigonometric Ratios:



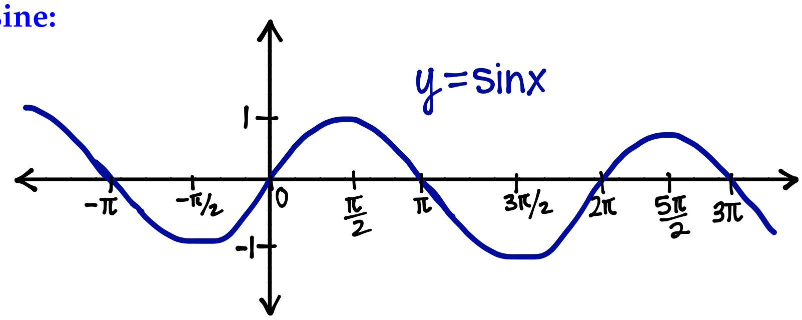
$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} = \frac{y}{r} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$$

$$\cos\theta = \frac{\text{adj.}}{\text{hyp.}} = \frac{x}{r} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$$

$$\tan\theta = \frac{\text{opp.}}{\text{adj.}} = \frac{y}{x} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$$

Basic Trigonometric Functions

sine:



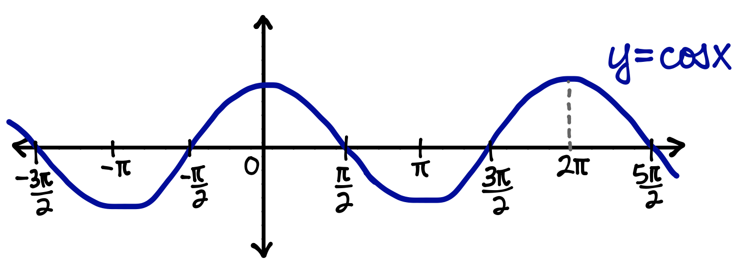
domain: $(-\infty,\infty)$

range: $[-1,1]$

period: 2π

roots: at $x=k\pi$, $k\in\mathbb{Z}$ (K is an integer)

cosine:



domain: $(-\infty,\infty)$

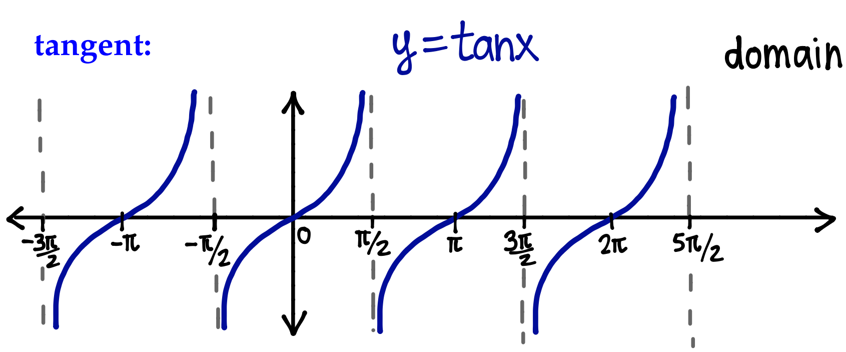
range: $[-1,1]$

period: 2π

roots: at $x=(2k+1)\cdot\frac{\pi}{2}$, $k\in\mathbb{Z}$

(odd multiples of $\frac{\pi}{2}$)

tangent:



domain: $\{x\in\mathbb{R} : x\neq(2k+1)\frac{\pi}{2}, k\in\mathbb{Z}\}$

(Vertical asymptotes at all odd integer multiples of $\frac{\pi}{2}$)

range: $(-\infty,\infty)$ period: π

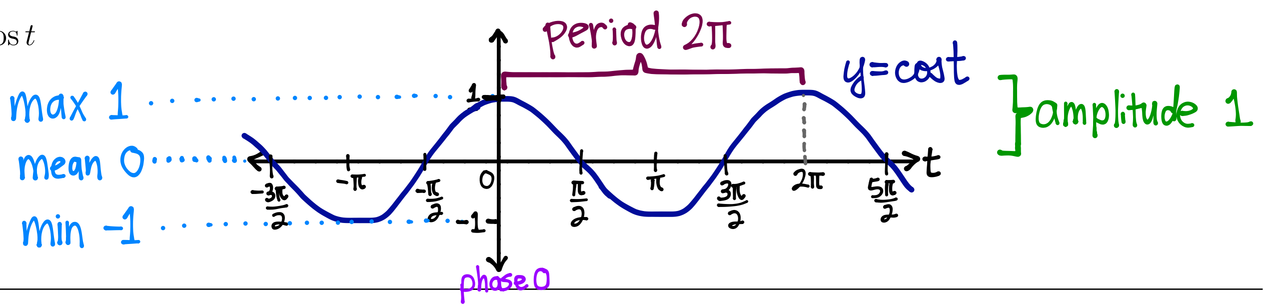
roots: at $x=k\pi$, $k\in\mathbb{Z}$

Useful Trig Identity

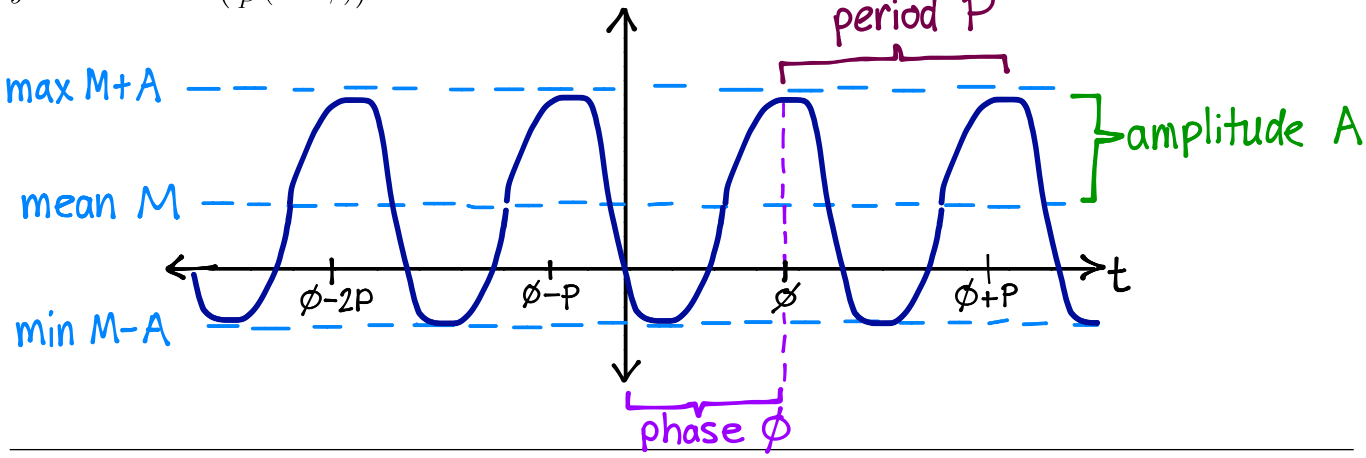
$\cos^2 x + \sin^2 x = 1$ (other identities later)

Oscillating Functions — Transformations of Cosine

$y = \cos t$



$y = M + A\cos\left(\frac{2\pi}{P}(t - \phi)\right)$



period b**:** $y = \cos\left(\frac{2\pi}{P}t\right)$ is a horizontal scaling of $y = \cos t$ by a factor of $\frac{P}{2\pi}$

$\Rightarrow$ period is scaled horizontally from 2π to $(2\pi)\left(\frac{P}{2\pi}\right) = P$ period $= \frac{2\pi}{b}$

phase ϕ**:** $y = \cos\left(\frac{2\pi}{P}(t - \phi)\right)$ is a horizontal shift of $y = \cos\left(\frac{2\pi}{P}t\right)$ by ϕ units right

$\Rightarrow$ phase shifts horizontally from 0 to $0 + \phi = \phi$.

amplitude A**:**

$y = A\cos\left(\frac{2\pi}{P}(t - \phi)\right)$ is a vertical scaling of $y = \cos\left(\frac{2\pi}{P}(t - \phi)\right)$ by a factor of A.

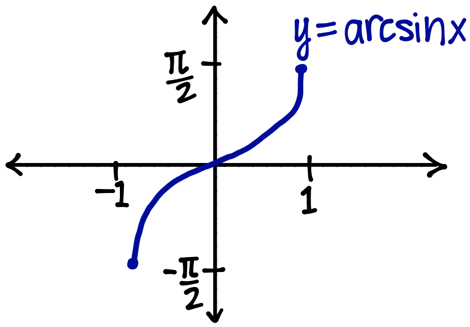
$\Rightarrow$ amplitude is scaled vertically from 1 to $(1)(A) = A$

mean M**:**

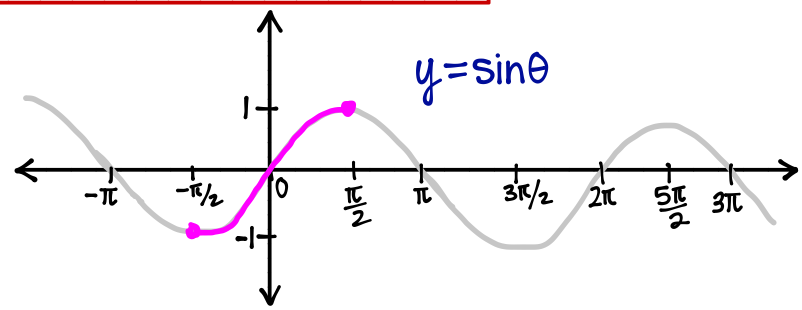
$y = M + A\cos\left(\frac{2\pi}{P}(t - \phi)\right)$ is a vertical shift of $y = A\cos\left(\frac{2\pi}{P}(t - \phi)\right)$ up M units

$\Rightarrow$ mean shifts vertically from 0 to $0 + M = M$, max $= M + A$, and min $= M - a$

arcsine: $\arcsin x = \theta \iff \sin\theta = x$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

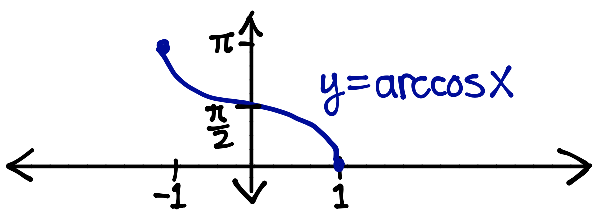


domain: $[-1,1]$ range: $[-\pi/2, \pi/2]$



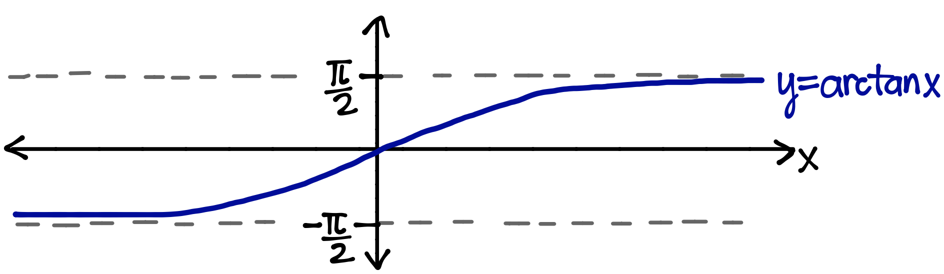
$y=\sin\theta$ is not one-to-one (fails H.L.T.)
To fix this issue, we restrict to a representative one-to-one chunk of $y=\sin\theta$

arccosine: $\arccos x = \theta \iff \cos\theta = x$ and $0 \leq \theta \leq \pi$ ← same idea as with $\sin\theta$ and $\arcsin x$



domain: $[-1,1]$ range: $[0,\pi]$

arctangent: $\arctan x = \theta \iff \tan\theta = x$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ ← same idea as with $\sin\theta$ and $\arcsin x$



domain: $(-\infty,\infty)$ range: $(-\frac{\pi}{2},\frac{\pi}{2})$

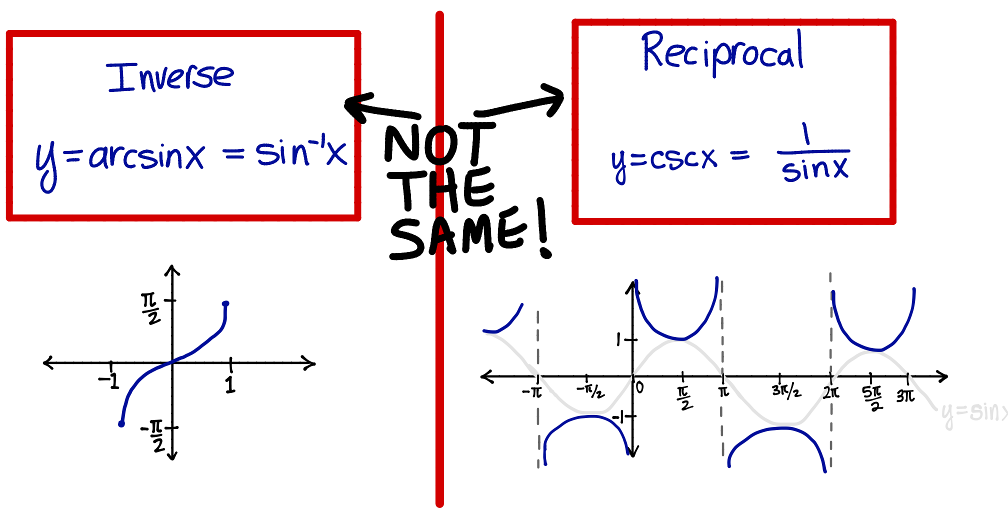
Beware of Confusing Notation for Inverse Trig Functions vs. Reciprocal Trig Functions !

- For a function f, we write f^{-1} for the inverse of f, which is NOT the same as the reciprocal $\frac{1}{f}$

Ex. The inverse of the sine function is denoted $\sin^{-1}(x)$ or $\arcsin(x)$.

- Reciprocal trig functions have their own names and are NOT written using the power -1

Ex. The reciprocal of the sine function is called the cosecant function and it's denoted $\csc(x)$



$\cot(x) \overset{\checkmark}{=} \frac{1}{\tan(x)}$ but $\frac{1}{\tan(x)} \neq \tan^{-1}(x) \overset{\checkmark}{=} \arctan x$

$\sec(x) \overset{\checkmark}{=} \frac{1}{\cos(x)}$ but $\frac{1}{\cos(x)} \neq \cos^{-1}(x) \overset{\checkmark}{=} \arccos x$

$\csc(x) \overset{\checkmark}{=} \frac{1}{\sin(x)}$ but $\frac{1}{\sin(x)} \neq \sin^{-1}(x) \overset{\checkmark}{=} \arcsin x$

Study Guide & Exercises

Adler & Lovrić, 2nd ed.	§1.4 pg. 44 # 13–20, 25, 27, 33–37, 39, 41 §2.2, pg. 89 # 23–26, 31–41, 57, 59–61, 63, 65 §2.3, pg. 107 # 45, 49, 64–71
Course Guide (on Brightspace)	§1.1 pg. 3 # 5, 11, 12 §2.1 pg. 6 # 1–7
DGD Workbook (on Brightspace)	pg. 2–6: LEC 1 & LEC 2