EIGENVALUES AND EIGENVECTORS

Def'n An eigenvector for $A_{n\times n}$ with associated eigenvalue satisfies the condition:

$$A\vec{x} = \lambda\vec{x}$$

where λ is the eigenvalue, $\vec{x}$ is the eigenvector, and we may refer to the associated λ & $\vec{x}$ as an eigen-pair.

Ex If $A=\begin{pmatrix}3 & -2\\ 1 & 0\end{pmatrix}$, $\vec{x}=\begin{pmatrix}2\\1\end{pmatrix}$, $\lambda=2$, then show that $\{\lambda,\vec{x}\}$ are an eigen-pair of A.

$LS = A\vec{x}$

$= \begin{pmatrix}3 & -2\\ 1 & 0\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}$

$= \begin{pmatrix}4\\2\end{pmatrix} = RS$ ✓

$RS = \lambda\vec{x}$

$= (2)\begin{pmatrix}2\\1\end{pmatrix}$

$= \begin{pmatrix}4\\2\end{pmatrix}$

Ex Is $\vec{x}=\begin{pmatrix}0\\1\\3\end{pmatrix}$ a valid eigenvector of $B=\begin{pmatrix}2 & 0 & 0\\ 1 & 2 & -1\\ 1 & 3 & -2\end{pmatrix}$

$B\vec{x}=\begin{pmatrix}2 & 0 & 0\\ 1 & 2 & -1\\ 1 & 3 & -2\end{pmatrix}\begin{pmatrix}0\\1\\3\end{pmatrix}=\begin{pmatrix}0\\-1\\-3\end{pmatrix}= \ldots\ldots\ldots = \lambda\begin{pmatrix}0\\1\\3\end{pmatrix}=\lambda\vec{x}$

guess?

Here we can see $\lambda=-1$ is valid, so the eigen-pair is $\left\{-1, \begin{pmatrix}0\\1\\3\end{pmatrix}\right\}$. So $\vec{x}$ is valid.

Counter Example: Is $\vec{x}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ an eigenvector of B?

$B\vec{x}=\begin{pmatrix}2 & 0 & 0\\ 1 & 2 & -1\\ 1 & 3 & -2\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\2\\3\end{pmatrix}=\ldots=\lambda\begin{pmatrix}0\\1\\0\end{pmatrix}$

guess?

* No such λ exists that satisfies the condition.

CHARACTERISTIC POLYNOMIAL

Def'n λ is an eigenvalue of $A_{n \times n}$ iff λ satisfies $c_A(\lambda) = |A - \lambda I| = 0$, where $c_A(\lambda)$ is called the characteristic polynomial.

Ex Find the characteristic polynomial of $A = \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix}$

$$0 = |A - \lambda I| = \begin{vmatrix} 3-\lambda & 5 \\ 1 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) - (5)(1)$$

$$= -3 - 3\lambda + \lambda + \lambda^2 - 5 = \lambda^2 - 2\lambda - 8 = c_A(\lambda)$$

Follow up: What are the eigenvalues?

$$0 = c_A(\lambda) = \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2)$$

$$\therefore \lambda = 4, -2$$

Ex Find $c_B(\lambda) = 0$, and the eigenvalues of $B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{pmatrix}$

$$0 = \det(B - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & -1 \\ 1 & 3 & -2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\Big((2-\lambda)(-2-\lambda) - (-1)(3)\Big) = (2-\lambda)\left(-4 - 2\lambda + 2\lambda + \lambda^2 + 3\right)$$

$$= (2-\lambda)(\lambda^2 - 1) = c_B(\lambda)$$

Now, $(2-\lambda)(\lambda^2 - 1) = (2-\lambda)(\lambda+1)(\lambda-1)$

$$\therefore \lambda = 2, 1, -1$$

Ex Find $C_C(\lambda)=0$, and the eigenvalues of

$$C=\begin{pmatrix} 5 & -2 & 6 & 1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$0=\begin{vmatrix} 5-\lambda & -2 & 6 & 1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix}=(5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

$\therefore\ \lambda = 5, 3, 5, 1$

NOTE: ① For $A_{n\times n}$ which is (WLOG) upper-triangular, the eigenvalues are the values on the main diagonal.

② For any set of λ's given a characteristic polynomial, if there are repeat eigenvalues, we should list the both/all.

③ Th'm Given $A_{n\times n}$ with eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, $\det(A)=(\lambda_1)(\lambda_2)\ldots(\lambda_n)$

④ Corollary If $\lambda_i=0$, then $\det(A)=0$, thus is not invertible.

Ex Find $C_D(\lambda)=0$, and the eigenvalues of

$$D=\begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$$

$$0=\begin{vmatrix} 3-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda)-(-5)(1) = -3-3\lambda+\lambda+\lambda^2+5$$

$$= \lambda^2-2\lambda+2$$

$$\therefore \lambda = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(2)}}{2(1)} = \frac{2\pm\sqrt{-4}}{2} = 1\pm i$$

$$\lambda = 1+i,\ 1-i$$

EIGENVECTOR

Recall $B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{pmatrix}$, $\lambda = 2, 1, -1$

The eigenvector solves the homogeneous system $(B - \lambda I)\vec{x} = \vec{0}$. The vectors "form a basis for the associated eigen-space", or more generally, "a basis for the null-space."

First, $\left(B - \lambda I\right) = \begin{pmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & -1 \\ 1 & 3 & -2-\lambda \end{pmatrix}$

$\lambda = 2$ $\left(\begin{array}{ccc|c} 2-(2) & 0 & 0 & 0 \\ 1 & 2-(2) & -1 & 0 \\ 1 & 3 & -2-(2) & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 3 & -4 & 0 \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\begin{cases} x - z = 0 \\ y - z = 0 \\ \hline z = t \end{cases}$

$= \begin{cases} x = t \\ y = t \\ z = t \end{cases} = \left\{ \vec{x} = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right.$ $\therefore$ let $t = 1$, $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Check

$Bx = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ ✓ $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = (2)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \lambda x$

$\lambda = 1$ $\begin{pmatrix} 2-(1) & 0 & 0 \\ 1 & 2-(1) & -1 \\ 1 & 3 & -2-(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$$\begin{cases} x = 0 \\ y - z = 0 \\ \hline z = t \end{cases} = \begin{cases} x = 0 \\ y = t \\ z = t \end{cases} = \left\{ \vec{x} = t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right. \quad \therefore \text{let } t = 1, \quad \vec{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = -1$ $\begin{pmatrix} 2-(-1) & 0 & 0 \\ 1 & 2-(-1) & -1 \\ 1 & 3 & -2-(-1) \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & -1 \\ 1 & 3 & -1 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix} \begin{cases} x = 0 \\ y - \frac{1}{3}z = 0 \\ \hline z = t \end{cases} = \begin{cases} x = 0 \\ y = \frac{1}{3}t \\ z = t \end{cases}$$

$$= \left\{ \vec{x} = t \begin{pmatrix} 0 \\ 1/3 \\ 1 \end{pmatrix} \right. \qquad \therefore \text{let } t = 3, \quad \vec{x} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

NOTE: ① If λ is an eigenvalue of the matrix, then there exists at least one eigenvector.

② Thus, when solving for a given eigenvector, we are guaranteed at least one row of zeros in RREF form.

③ WARNING: If $\lambda \in \mathbb{C}$, then the row operations, vec. param. sol'n, and property $A\vec{x} = \lambda\vec{x}$ may not be super clean → BE CAREFUL

Ex Recall $D = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$, $\lambda = 1+i, 1-i$

First, $(D - \lambda I) = \begin{pmatrix} 3-\lambda & -5 \\ 1 & -1-\lambda \end{pmatrix}$

$\lambda = 1+i$ $\begin{pmatrix} 3-(1+i) & -5 \\ 1 & -1-(1+i) \end{pmatrix} = \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}$

$$\begin{pmatrix} 2-i & -5 \\ 2-i & -4+i^2 \end{pmatrix} (2-i)R_2 = \begin{pmatrix} 2-i & -5 \\ 2-i & -5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2-i & -5 \\ 0 & 0 \end{pmatrix} R_2 - R_1 \sim \begin{pmatrix} \frac{2-i}{2-i} & \frac{-5}{2-i} \\ 0 & 0 \end{pmatrix} \frac{1}{2-i} R_1$$

$$= \begin{pmatrix} 1 & -2-i \\ 0 & 0 \end{pmatrix}$$

* Rationalize the denominator

$$\frac{-5}{2-i} \cdot \frac{2+i}{2+i} = \frac{-10-5i}{5} = -2-i$$

$$\left\{ \frac{x + (-2-i)y = 0}{y = t} \right. = \left\{ \begin{array}{l} x = (2+i)t \\ y = t \end{array} \right. = \left\{ \vec{x} = t \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \right.$$

$\therefore$ Let $t = 1$, $\vec{x} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$

OR/ Let $t = 2-i$ (the conjugate of $2+i$, the x-component)

$$\vec{x} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$$

OR/ Then divide by 5: $\vec{x} = \begin{pmatrix} 1 \\ \frac{2}{5} - \frac{1}{5}i \end{pmatrix}$

$\lambda = 1-i$ $\begin{pmatrix} 3-(1-i) & -5 \\ 1 & -1-(1-i) \end{pmatrix} = \begin{pmatrix} 2+i & -5 \\ 1 & -2+i \end{pmatrix}$

$\sim \begin{pmatrix} 1 & -2+i \\ 0 & 0 \end{pmatrix}$

* Since we know that after RREF we are guanteed a row of zeros.

$\left\{ \begin{array}{l} x + (-2+i)y = 0 \\ \hline y = t \end{array} \right. = \left\{ \begin{array}{l} x = (2-i)t \\ y = t \end{array} \right. = \left\{ \vec{x} = t\begin{pmatrix} 2-i \\ 1 \end{pmatrix} \right.$

$\therefore$ Let $t = 1$, $\vec{x} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$

or/ Let $t = 2+i$, $\vec{x} = \begin{pmatrix} 5 \\ 2+i \end{pmatrix}$

or/ ... $\vec{x} = \begin{pmatrix} 1 \\ \frac{2}{5} + \frac{1}{5}i \end{pmatrix}$

DIAGONALIZATION

If a matrix is diagonalizable, then A can be expressed as $A = PDP^{-1}$, where D is a diagonal matrix (with ordered eigenvalues), and P is a [square, invertible] matrix (with ordered eigenvector-columns).

Ex Recall $B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{pmatrix}$

$\lambda = 2,\ \vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $\lambda = 1,\ \vec{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$;

$\lambda = -1,\ \vec{x} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$

$\therefore B = PDP^{-1}$ where $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

OR) where $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 1 \end{pmatrix}$

EVALUATING POWERS

If $A = PDP^{-1}$, then $A^{\alpha} \overset{*}{=} (PDP^{-1})^{\alpha} = \underbrace{(PDP^{-1})(PDP^{-1})\dots(PDP^{-1})}_{\alpha\text{-times}}$

$$A^{\alpha} = P D^{\alpha} P^{-1}$$

Recall: Given D is diagonal, then

$$D^{\alpha} = \begin{pmatrix} (d_{11})^{\alpha} & & & \\ & (d_{22})^{\alpha} & & 0 \\ & & \ddots & \\ 0 & & & (d_{nn})^{\alpha} \end{pmatrix}$$

<u>Ex</u> Let $A = PDP^{-1}$, where $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$

1/ Evaluate: A^{10}

First, $P^{-1} = \frac{1}{3-2}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

Now, $A^{10} = PD^{10}P^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{10}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \ldots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

2/ Evaluate: $A^{9} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{9}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -4 \\ 6 & -5 \end{pmatrix}$$

<u>Ex</u> Diagonalize $A = \begin{pmatrix} 6 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

So, given $A = PDP^{-1}$, find a diagonal matrix D, and invertible matrix P.

From tutorial we have: $\lambda_1 = 6$, $\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$;

$\lambda_2 = 3$, $\vec{x} = \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix}$; $\lambda_3 = 1$, $\vec{x} = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}$

$\therefore$ $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 5 \\ 0 & -3 & 0 \end{pmatrix}$

Ex Diagonalize $C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ by finding a matrix D (diagonal) and P (invertible), s.t. $C = PDP^{-1}$.

We are given $C_c(\lambda) = (2-\lambda)(\lambda+1)(\lambda+1) = 0$

So $\lambda = 2, -1, -1$

NOTE: ① $\lambda = -1$ is a repeat eigenvalue. It appers twice, so we list it twice. Thus,

Def'n Algebraic Multiplicity of an eigenvalue is an expression for how many times an eigenvalue repeats.

e.g. For $\lambda = -1$ Alg. multip. $= 2$

$\lambda = -1$ // $\begin{pmatrix} 0-(-1) & 1 & 1 \\ 1 & 0-(-1) & 1 \\ 1 & 1 & 0-(-1) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\left\{ \begin{array}{l} x + y + z = 0 \\ \hline y = s \\ z = t \end{array} \right. = \left\{ \begin{array}{l} x = -s - t \\ y = s \\ z = t \end{array} \right. = \left\{ \vec{x} = s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right.$

$\therefore$ Let $s = 1, t = 1$, then $\vec{x} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

NOTE: ② For $\lambda = -1$, the "basis of the eigenspace" is 2-dimensional: $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

③ Def'n Geometric Multiplicity of an eigenvector/eigenvalue is an expression for how many separate eigenvectors we have for a given eigenvalue.

e.g. for $\lambda=-1$, $\vec{x}=\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$, Geo. multip $=2$ $\checkmark$!2

$\lambda=2$

$$\begin{pmatrix}0-(2) & 1 & 1\\ 1 & 0-(2) & 1\\ 1 & 1 & 0-(2)\end{pmatrix}=\begin{pmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{pmatrix}\sim\begin{pmatrix}-2 & 1 & 1\\ 2 & -4 & 2\\ 2 & 2 & -4\end{pmatrix}$$

$$\sim\begin{pmatrix}-2 & 1 & 1\\ 0 & -3 & 3\\ 0 & 3 & -3\end{pmatrix}\sim\begin{pmatrix}-2 & 0 & 2\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix}\sim\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix}$$

$$\left\{\begin{array}{l}x-z=0\\ y-z=0\\ \hline z=t\end{array}\right. = \left\{\begin{array}{l}x=t\\ y=t\\ z=t\end{array}\right. = \left\{\vec{x}=t\begin{pmatrix}1\\1\\1\end{pmatrix}\right.$$

$\therefore$ Let $t=1$, $\vec{x}=\begin{pmatrix}1\\1\\1\end{pmatrix}$

Now, if $C=PDP^{-1}$, then $D=\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2\end{pmatrix}$

$$P=\begin{pmatrix}-1 & -1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}$$

NOTE: (4) For the above example, we can be asked "What is the dimension of the eigen-space?" In this case: $\dim\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}=2$, which is the same as asking for the Geometric Multiplicity.

Th'm Given $A_{n\times n}$, then $A=PDP^{-1}$ (that is A is diagonalizable) is only true iff for each eigenvalue, alg. multip. = geo. multip.

e.g. For $C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ ① $\lambda = -1$: Alg. multip = 2 } SAME 13
Geo. multip = 2

② $\lambda = 2$: Alg. multip. = 1 } SAME
Geo. multip. = 1

Therefore C is diagonalizable.

Th'm Given $A_{n\times n}$, if A has n-distinct eigenvalues then A is diagonalizable.

e.g. for $A = \begin{pmatrix} 6 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ $\lambda = 6, 1, 3$

Since those eigenvalue are distinct, we know A is diagonalizable.

REMINDER: If λ is an eigenvalue of a matrix, then we are guaranteed at least one eigenvector.

NOTE: ③ Geometric Multiplicity can never be greater than Algebraic Multiplicity.

Ex Is $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ diagonalizable?

① $0 = |X - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 \quad \therefore \underbrace{\lambda = 1, 1}_{\text{alg. multip} = 2}$

② $\lambda = 1$ $\begin{pmatrix} 1-(1) & 1 \\ 0 & 1-(1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\left\{ \dfrac{y=0}{x=t} \right. = \left\{ \vec{x} = t\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right.$

$\therefore$ Let $t=1$, $\underbrace{\vec{x} = \begin{pmatrix}1\\0\end{pmatrix}}_{\text{geo. multip} = 1}$

So X is NOT diagonalizable since for $\lambda = 1$

$2 = \text{alg. multip.} \neq \text{geo. multip.} = 1$

Here are some possible WRONG answers previously submitted by students:

1) $D = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

$P = \begin{pmatrix}1\\0\end{pmatrix}$ ← This is the only eigenvector; P is non-square, so this P matrix is invalid.

2) $D = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

$P = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$ ← Here the student gave $\begin{pmatrix}0\\0\end{pmatrix}$ as the other eigenvector; $\vec{0}$ is the trivial solution, thus it is not appropriate; this P matrix is not invertible

3) $D = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

$P = \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}$ ← Here the student gave the same eigenvector twice; P is still not invertible

4) $D = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$ ← This D matrix is now incorrect because it shows that the eigenvalues are $\lambda = 0, 1$ ... which is false

$P = \begin{pmatrix}* & *\\ * & *\end{pmatrix}$