MATRIX INVERSES

Consider the eq'n $ax = b$

To solve for x: $\frac{ax}{a} = \frac{b}{a}$

$x = \frac{b}{a}$

In fact, $a^{-1} = \frac{1}{a}$, which is the inverse [element] of a. The true operation was:

$$a^{-1}(ax) = a^{-1}(b)$$
$$(a^{-1}a)x = a^{-1}b$$
$$(1)x = a^{-1}b$$
$$x = \frac{b}{a}$$

NOTE: ① $a^{-1} \cdot a = 1$, likewise $a \cdot a^{-1} = 1$.

② In this case, 1 is the identity [element]

For matrices, $A_{n \times n}$, and suppose A^{-1} exists, then:

$$A^{-1}A = I, \text{ likewise } AA^{-1} = I$$

③ Given an A, if A^{-1} exists, then we also say it is unique.

Now, the Matrix Eq'n: $A\vec{x} = \vec{b}$

$$(A^{-1}A)\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

Here we will solve systems with A^{-1}!!! ... seen later.

METHOD FOR A 2×2

Sometimes referred to as the determinant-adjoint method:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then $A^{-1} = \dfrac{1}{\underbrace{ad - bc}_{\text{product of: major diagonal} - \text{minor diagonal}}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

- swap a & d
- negate b & c

Terminology: $A^{-1} = \dfrac{1}{\underbrace{\det A}_{\text{determinant of } A}} \operatorname{adj} A$ } adjoint of A

Ex Find A^{-1} if $A = \begin{pmatrix} 2 & 4 \\ -3 & 8 \end{pmatrix}$

$$\therefore A^{-1} = \frac{1}{(2)(8) - (-3)(4)} \begin{pmatrix} 8 & -4 \\ +3 & 2 \end{pmatrix}$$

$$= \frac{1}{28} \begin{pmatrix} 8 & -4 \\ 3 & 2 \end{pmatrix}$$

Check $A^{-1}A = \frac{1}{28}\begin{pmatrix} 8 & -4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -3 & 8 \end{pmatrix} = \frac{1}{28}\begin{pmatrix} 28 & 0 \\ 0 & 28 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ ✓

Ex Let $B = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$. Find B^{-1}

$$\therefore B^{-1} = \frac{1}{(1)(5)-(3)(2)} \begin{pmatrix} 5 & -3 \\ -2 & 1 \end{pmatrix} = \frac{1}{-1}\begin{pmatrix} 5 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$$

Check $BB^{-1} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Ex Is C invertible? Try to find C^{-1} for

$C = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ $\therefore C^{-1} = \frac{1}{(1)(6)-(2)(3)} \begin{pmatrix} 6 & -2 \\ -3 & 1 \end{pmatrix}$

$$= \frac{1}{0}\begin{pmatrix} 6 & -2 \\ -3 & 1 \end{pmatrix} = \text{DNE}$$

Def'n A matrix which can not be inverted is called singular. Likewise, if the inverse exists, we can say a matrix is non-singular, where the determinant is non-zero.

METHOD FOR A 3x3 (or bigger)

Sometimes referred to as the RREF method:

$$(A \,\vdots\, I) \sim (I \,\vdots\, A^{-1})$$

Here we assume a perfect RREF to a unique soln.

Ex Find the inverse of $D = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 1/3 \\ 0 & 2 & 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1/2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1/3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \sim \begin{array}{c} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1/3 & 0 & 1 & 0 \end{array}\right) \end{array} \begin{array}{l} 2R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 1/3 & 0 & 1 & 0 \end{array}\right) \tfrac{1}{2}R_2 \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/3 & 0 & 1 & -1/2 \end{array}\right) R_3 - R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 3 & -3/2 \end{array}\right) 3R_3 \qquad \therefore D^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 3 & -3/2 \end{pmatrix}$$

Check

$$DD^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 1/3 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 3 & -3/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

Ex Find the inverse of $\begin{pmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 0 & 1 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \end{array}\right) \begin{array}{l} R_3 \\ \\ R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 & 0 & -2 \end{array}\right) \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & -3 & 4 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 & -1 & -1 \end{array}\right) \begin{array}{l} R_1 - 3R_2 \\ \\ R_3 - R_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & -3 & 4 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}\right) \begin{array}{l} \\ \\ \frac{1}{2}R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/2 & -3/2 & 11/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & -1/2 \end{array}\right) \begin{array}{l} R_1 - 3R_3 \\ R_2 + R_3 \\ \\ \end{array}$$

So the inverse is:

$$\frac{1}{2}\begin{pmatrix} -3 & 3 & 11 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{pmatrix}$$

Check

$$\frac{1}{2}\begin{pmatrix} -3 & 3 & 11 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

PROPERTIES WITH INVERSES

Recall, A, B, C as nice matrices:

$$(ABC)^T = C^T B^T A^T$$

Now, with inverses:

(12) $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$

(13) $(A^{-1})^T = (A^T)^{-1} = A^{-T}$

(14) $\begin{cases} A^T A^{-T} = I \\ A^{-T} A^T = I \end{cases}$

(15) $(ABC)^{-T} = A^{-T} B^{-T} C^{-T}$

since $(ABC)^{-T} = ((ABC)^{-1})^T = (C^{-1}B^{-1}A^{-1})^T =$

$= (A^{-1})^T (B^{-1})^T (C^{-1})^T = A^{-T} B^{-T} C^{-T}$

Ex Simplify: $(AB)^T (ABC)^{-T}$

$= B^T A^T \; A^{-T} B^{-T} C^{-T}$

$= B^T (A^T A^{-T}) B^{-T} C^{-T}$

$= B^T (I) B^{-T} C^{-T}$

$= (B^T B^{-T}) C^{-T}$

$= (I) C^{-T} = C^{-T}$