MAT 3378 (Winter 2010)
Final Examination - Solutions

Spring 2010 **Professor G. Lamothe**
Duration: 3 hours

Student Number:____________

Last Name: ____________ **First Name:** ____________

- **This is a closed book examination.**
- **Only non-programmable and non-graphic calculators are permitted.**
- **Two sheets double sided are permitted.**
- **There are six questions.**

1. [**10 points**] A canning plant uses a large number of machines for the filling process. Each machine is supposed to pour a specified weight of product into each container. The plant manager suspects that there is too much variation in weight of the product poured among the machines. To check this suspicion, the plant manager selects four machines at **random** and weighs the contents of five randomly selected containers filled by each of the four randomly selected machines.

   a) Write down the corresponding one-factor ANOVA model.

   b) Complete the following ANOVA table to test for machine effects. Give the null and alternative hypotheses and the value of the corresponding test statistic.

source	df	SS	MS	F	p-value
machine	3		0.0015650		< 0.0001
error		0.0012			
total	19				

   c) What are the appropriate procedures to describe the effects in the context of this problem? **Note:** Describe the techniques that should be used to describe the effects. You do not need to describe the effects.

   d) How much of the variance in the weights is attributed to differences among the machines?

   Solution:
   a) The appropriate model is a one factor ANOVA model with random effects:

$$Y_{ij} = \mu_i + \varepsilon_{ij},$$

   where

$$\mu_i \text{ are independent } N(\mu, \sigma_\mu^2); \quad \varepsilon_{ij} \text{ are independent } N(\mu, \sigma^2);$$

$$\mu_i, \varepsilon_{ij} \text{ are independent}$$

   for $i = 1, \ldots 4$ and $j = 1, \ldots, 5$.

   b)

source	df	SS	MS	F	p-value
machine	3	.00465	0.0015650	20.87	< 0.0001
error	16	0.0012			
total	19	.005895			

The hypotheses for testing for treatment effects are

$$H_0 : \sigma_\mu^2 = 0 \quad \text{against} \quad H_a : \sigma_\mu^2 \neq 0.$$

The observed value of the test statistic is $F^* = \text{MSTR}/\text{MSE} = 20.87$.

c) Estimate the variance of the random effect σ_μ^2, but more importantly estimate the intra-class correlation which gives the proportion of the variance in the response which is explained by the variance of the treatment effect.

d) $s_\mu^2 = (\text{MSRT} - \text{MSE})/n = 0.000298$ and so the estimated intra-class correlation is

$$\widehat{\text{ICC}} = \frac{s_\mu^2}{s_\mu^2 + \text{MSE}} = 79.9\%.$$

2. **[10 points]** Consider the following data for a randomized complete block design study where block and treatment effects are fixed.

	Treatment		
Block	$j=1$	$j=2$	$j=3$
$i=1$	11.5	11.8	12.9
$i=2$	15.3	13.7	16.2
$i=3$	18.3	21.3	23

Here are few corresponding sums:

$$\mathsf{SSBL} = 120.08, \mathsf{SSTR} = 8.8867, \mathsf{SSTO} = 135.7$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3}(\overline{Y}_{i\cdot} - \overline{Y}_{\cdot\cdot})(\overline{Y}_{\cdot j} - \overline{Y}_{\cdot\cdot})Y_{ij} = 16.6211.$$

(a) For such an ANOVA model, we do not incorporate an interaction term $(\rho\tau)_{ij}$. What problem are we going to encounter if we do add the interaction term $(\rho\tau)_{ij}$ into the model?

(b) Perform the Tukey test for additivity. Give the null and alternative hypotheses and compute the observed value of the test statistic.

(c) Using $\alpha = 5\%$ and the following quantiles from the F distribution

$$F(0.95; 1, 3) = 10.13, \quad F(0.95; 1, 4) = 7.71, \quad F(0.95; 2, 4) = 6.94,$$

give the conclusion of the test from part (b).

(d) Construct a family of confidence interval of pairwise differences with a family coefficient of 95% to describe the treatment effects. You can use the following quantiles from the studentized range distribution:

$$q(0.95; 3, 4) = 5.04024, \ q(0.95; 6, 4) = 4.89560, \ q(0.9917; 3, 4) = 8.54930.$$

Solution:
a) If the mean response is

$$E\{Y_{ij}\} = \mu_{\cdot} + \rho_i + \tau_j + (\rho\tau)_{ij} + \varepsilon_{ij},$$

such that $\sum_i \rho_i = 0$, $\sum_j \tau_j = 0$, $\sum_i (\rho\tau)_{ij} = 0$, $\sum_j (\rho\tau)_{ij} = 0$, then there are $n_b\, r = 9$ free parameters to estimate. This means that there are no degrees of freedom left to estimate the variance of the random error.

b) The hypotheses for the Tukey test for additivity are

$$H_0 : D = 0 \quad \text{against} \quad H_a : D \neq 0,$$

where $(\rho\,\tau)_{ij} = D\,\rho_i\,\tau_j$.

The sum of squares for the special interaction is

$$SSBL.TR^* = \frac{(16.6211)^2}{(SSBL/3)(SSTR/3)} = 2.335.$$

The special remainder sum of squares is

$$\text{SSRem}^* = \mathsf{SSTO} - \mathsf{SSBL} - \mathsf{SSTR} - \mathsf{SSBL.TR}^* = 4.418.$$

The observed value of the test statistic is

$$F^* = \frac{\mathsf{SSBL.TR}^*}{\mathsf{SSRem}^*/(n_b\, r - n_b - r)} = \frac{2.335}{4.418/3} = 1.5856.$$

c) Since $F^* < F(.095; 1, 3)$, then we cannot reject H_0. There appears to be no interaction.

d) Using Tukey's method the form of the CI for $\mu_{\cdot j} - \mu_{\cdot j'}$ is

$$\overline{Y}_{\cdot j} - \overline{Y}_{\cdot j'} \pm \frac{1}{\sqrt{2}} q(.95; 3, 4)\sqrt{MSE}\sqrt{2/n_b} = \overline{Y}_{\cdot j} - \overline{Y}_{\cdot j'} \pm E,$$

where

$$E = \frac{1}{\sqrt{2}}(5.04024)\sqrt{\frac{135.7 - 120.08 - 8.867}{(3-1)(3-1)}}\sqrt{2/3} = 3.7810.$$

The family of confidence interval with a coefficient of confidence of 95% is:

$$\text{CI for } \mu_{\cdot 1} - \mu_{\cdot 2} : 15.033 - 15.6 \pm 3.7810 = -.567 \pm 3.7810;$$

$$\text{CI for } \mu_{\cdot 1} - \mu_{\cdot 3} : 15.033 - 17.367 \pm 3.7810 = -2.334 \pm 3.7810;$$

$$\text{CI for } \mu_{\cdot 1} - \mu_{\cdot 2} : 15.6 - 17.367 \pm 3.7810 = -1.767 \pm 3.7810.$$

3. **[10 points]** Consider the data from Question 2. Now suppose that each block represents a subject that was asked to rate three soft drinks (the treatments) by given a number between 0 and 25.

(a) Write down the appropriate model for this study with all assumptions.

(b) For such a study we often assume the observations satisfy a compound symmetry. Explain what this means.

(c) Below is a matrix of the sample within subject between treatment correlations. What do these values suggest in terms of the underlying assumptions of our ANOVA model.

Pearson Correlation Coefficients, N = 3
Prob > |r| under H0: Rho=0

	_1	_2	_3
_1	1.00000	0.92056 0.2555	0.96502 0.1689
_2	0.92056 0.2555	1.00000	0.99077 0.0866
_3	0.96502 0.1689	0.99077 0.0866	1.00000

(d) For such a study there are potential disadvantages such as *order effect* and *carryover effect*. What are these effects? What steps can be taken to minimize these interfering effects?

(e) Test for treatment effects at $\alpha = 0.05$. Give the null and alternative hypotheses. You can use the following quantiles from the F distribution:

$$F(0.95; 1, 3) = 10.13, \quad F(0.95; 1, 4) = 7.71, \quad F(0.95; 2, 4) = 6.94.$$

Solution:
a) The model for this repeated measures design is:

$$Y_{ij} = \mu_{..} + \rho_i + \tau_j + \varepsilon_{ij},$$

such that ρ_i are independent $N(0, \sigma_\rho^2)$, ε_{ij} are independent $N(0, \sigma^2)$, ρ_i, ε_{ij} are independent, and $\sum_{j=1}^n \tau_j = 0$.

b) a common intra-subject correlation

c) Similar correlation between treatments suggests that the compound symmetry could be satisfied.

d) The **order effect** refers to the effect that the order might have on the response. By randomly assigning the order of the treatments within the subjects, then a particular treatment will not always be at the same position and thus averaging out the order effects.

The **carry over** effect refers to the effect that a treatment could have on future responses. Randomly assigning the treatments within a subject will minimize this effect since a treatment might not always follow the same particular treatment.

e)

$$F^* = \frac{\text{MSTR}}{\text{MSE}} = \frac{8.8867/2}{(135.7 - 20.08 - 8.867)/4} = 2.63.$$

Since $F^* < F(.95; 2, 4)$, then we should not reject H_0. It appears that there are no treatment effects.

4. **[10 points]** Three states (factor A) participated in a health awareness study. Each state independently devised a health awareness program. Three cities (factor B) from each state were randomly chosen for participation in the program and within each city five households were chosen randomly. All members of the selected household were interviewed before and after participation in the program and a composite index was formed for each household measuring the impact of the health awareness program. The data on health awareness follow (the larger the index, the greater the awareness).

	state i:	1			2			3		
	city j:	1	2	3	1	2	3	1	2	3
Household	$k = 1$:	42	26	34	47	56	68	19	18	16
Household	$k = 2$:	56	38	51	58	43	51	36	40	28
Household	$k = 3$:	35	42	60	39	65	49	24	27	45
Household	$k = 4$:	40	35	29	62	70	71	12	31	30
Household	$k = 5$:	28	53	44	65	59	57	33	23	21

(a) The appropriate model in this case is a nested design with factor B nested in factor A where factor A has fixed effects and factor B has random effects. Write down the corresponding ANOVA model.

(b) Using the following SAS program, we obtained the within state ANOVA tables.

```
proc glm data=nested;
by state;
class city;
model awareness = city;
run;
```

Here are the three within state ANOVA tables:

State 1

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	60.933333	30.466667	0.25	0.7810
Error	12	1448.800000	120.733333		
Corrected Total	14	1509.733333			

State 2

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	74.533333	37.266667	0.35	0.7137
Error	12	1288.800000	107.400000		
Corrected Total	14	1363.333333			

State 3

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	32.133333	16.066667	0.17	0.8483
Error	12	1155.600000	96.300000		
Corrected Total	14	1187.733333			

Using the above tables to complete the following ANOVA table.

source	df	SS	MS	F
state				32.26
city (within state)				0.26
total	44	11037.64444		

(c) Using the following SAS program, we produced the table of expected mean squares found below.

```
proc glm data=nested;
class state city ;
model awareness = state city(state);
random city(state);
run;
```

Source	Type III Expected Mean Square
state	Var(Error) + 5 Var(city(state)) + Q(state)
city(state)	Var(Error) + 5 Var(city(state))

Give the null and alternative hypotheses for testing for state main effects. Compute the observed value of the test statistic and give its sampling distribution under H_0.

(d) Estimate the variance component σ^2_β and compute the proportion of the variance in the response explained by the variance of the city effects.

Solution:
a)

$$Y_{ijk} = \mu_{..} + \alpha_i + \beta_{j(i)} + \varepsilon_{ijk},$$

where

$$\sum_i \alpha_i = 0,$$

$$\beta_{j(i)}, \varepsilon_{ijk} \text{ are independent},$$

$$\beta_{j(i)} \text{ are independent } N(0, \sigma^2_\beta),$$

$$\alpha_{j(i)} \text{ are independent } N(0, \sigma^2_\alpha).$$

b) $SSB(A) = \sum_i SSB(A_i) = 167.599996$ and adding the degrees of freedom gives 6 df for SSB(A).
$SSE = \sum_i SSE(A_i) = 3893.2$ and adding the degrees of freedom gives 36 df for SSE.
Lastly $SSA = SSTO - SSB(A) - SSE = 6976.84444$. Using the additivity of the degrees of freedom we get $44 - 36 - 6 = 2$ df for SSA.

source	df	SS	MS	F
state	2	6976.84444	3488.42222	32.26
city (within state)	6	167.599996	27.93333	0.26
error	36	3893.2		
total	44	11037.64444		

c) We are testing

$$H_0 : \alpha_1 = \alpha_2 = \alpha_3 = 0 \quad \text{against} \quad H_a : \text{not all zero.}$$

The observed value of the test statistic is

$$F^* = \frac{MSA}{MSB(A)} = \frac{3488.42222}{27.93333} = 124.88.$$

Under H_0, $F^* \sim F(2,6)$.

d)

$$s_\beta^2 = \frac{MSB(A) - MSE}{5} < 0.$$

So we will take $s_\beta^2 = 0$.

5. **[10 points]** In a study to investigate the effect of the colour of paper (blue, green, orange) on response rates for questionnaires distributed by the "windshield method" in supermarket parking lots, 15 representative supermarket parking lots were chosen in a metropolitan area and each colour was assigned at random to five of the lots. The response rates (in percent) follows.

blue	28	26	31	27	35
green	34	29	25	31	29
orange	31	25	27	29	28

(a) What type of study design is being implemented here?

(b) Using a one-factor ANOVA model to represent these data, we obtained the following ANOVA table.

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	7.6000000	3.8000000	0.39	0.6842
Error	12	116.4000000	9.7000000		
Corrected Total	14	124.0000000			

Assuming that the underlying assumptions of the ANOVA model hold, then conduct a test to determine whether or not the mean response rates for the three colors differ. State the null and alternative hypotheses and the conclusion at $\alpha = 5\%$.

(c) For the test in part b, the p-value is the probability of which event.

(d) When informed of the findings, an executive said: "See? I was right all along. We might as well print the questionnaires on plain white paper, which is cheaper." Does this conclusion follow from the findings of the study? Discuss.

(e) Consider the above study as a preliminary study. Consider the estimated parameters as the true values of the parameters of the above

ANOVA model. The power of the test for treatment effects is the probability of which event? *Hint:* Compute the value of the non-centrality parameter.

(f) Suppose that we would like to perform a similar study with $r = 6$ colours. We will use MSE from this study as an estimate of σ^2. Suppose that we would like to detect a range in the response rates of 2.5 percent with high probability. As a function of n=number of replicates, power for the test for treatment effects is the probability of which event?

Solution:
(a) completely randomized design

(b) We are testing

$$H_0 : \mu_1 = \mu_2 = \mu_3 \quad \text{against} \quad H_a : \text{ not all equal}$$

The p-value is 0.6842, which is larger than 5%. We cannot reject H_0. It appears that the means are equal.

c) The p-value is $P[F(2, 12) > 0.39]$.

d) Discuss the fact that white was not involved in the study.

e) Power is

$$P[F(2, 12, \text{nc}) > F(.95; 2, 12)],$$

where

$$\text{nc} = \sum_i \sum_j \frac{\alpha_i^2}{\sigma^2} = \frac{SSTR}{MSE} = 0.784$$

f) Power as a function of n is

$$P[F(5, 6(n-1), \text{nc}) > F(.95; 5, 6(n-1)],$$

where

$$\text{nc} \geq \frac{n\Delta^2}{2\,\sigma^2} = \frac{n(2.5)^2}{2(9.7)} = 0.322\,n.$$

We use the above lower bound of nc to get a conservative estimate of power.

6. [**10 points**] An appraisal service employs three appraisers for assessing the values of residential properties. The working assumption of the service manager is that the appraisals of the three appraisers are the same, on average. To check the assumption, the manager selects 15 residential properties that are regarded to be essentially the same in value and randomly assigns five of these properties to each appraiser. As a result, the following summary data are obtained, where appraisals are in thousands of dollars.

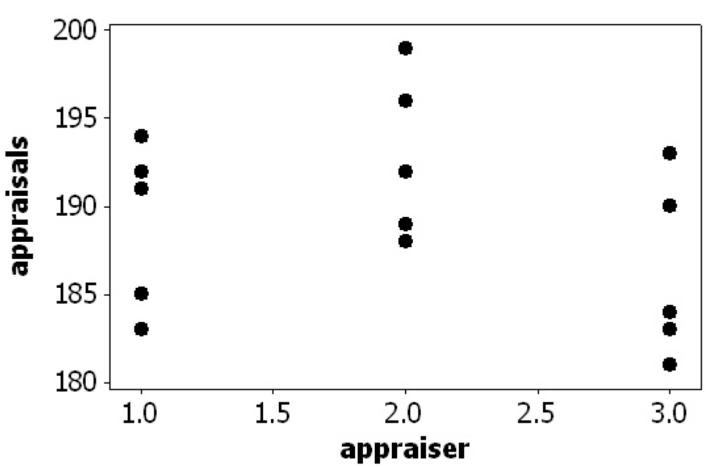
Here are within appraiser statistics:

Variable	sample size	Mean	StDev
appraiser 1	5	189.00	4.74
appraiser 2	5	192.80	4.66
appraiser 3	5	186.20	5.07

The overall statistics are

Variable	sample size	Mean	StDev
appraisals	15	189.33	5.27

The data are displayed in the following scatter plot.



a) Identify the sources of variation for this problem and based on the above scatter plot determine whether the appraisals of the three appraisers appear to be the same, on average.

b) Should we consider the appraiser effects as fixed or random? Discuss.

c) Compute the proportion of the total variability of the appraisals which is explained by the appraiser factor.

d) The p-value for the test for appraiser effects is about 0.14. Is this consistent with your investigations from part 1? Do you believe that the test lacks power or simply that there are no appraiser effects? Discuss.

e) We assumed that the 15 properties were essentially the same in value. Discuss an alternative experimental design to use in comparing the three appraisers that should be more powerful.

f) For the design that you proposed in part d, write down the corresponding ANOVA model.

Solution:
a) The appraisers are a source of variation. Based on the scatter plot, it does appear that the appraisers give different appraisals on average. In particular, appraiser 2 appears to give appraisals which are larger in value on average.

b) Since we want to compare the appraisals from these particular appraisers, then the appraiser effects should be considered as fixed.

c)

$$SSE = \sum_{i=1}^{n_i} (n_i - 1) s_i^2 = 279.5524,$$

$$SSTO = (n_T - 1)\, s^2 = 388.8206.$$

The proportion of the total variability of the appraisals which is explained by the appraiser factor is

$$R^2 = 1 - \frac{\text{SSE}}{\text{SSTO}} = 0.281 = 28.1\%.$$

d) There appear to be appraiser effect but we cannot reject the null hypothesis that there are no appraiser effects. It might be that the test is not powerful enough due to the small sample size or large unexplained variability. A cause of the large unexplained variability might be that the properties are actually heterogeneous, that is they might not all be valued exactly the same.

e) It might be difficult to find 15 properties of exactly the same value. It might be more reasonable to find 5 properties and ask the appraisers to evaluate all 5 properties.

f) It is a repeated measures design. The model is

$$Y_{ij} = \mu_{..} + \rho_i + \tau_j + \varepsilon_{ij},$$

where

$$\rho_i \text{ are independent } N(0, \sigma_\rho^2) \text{ [property effect]},$$

$$\sum_j \tau_j = 0,$$

and

$$\varepsilon_{ij} \text{ are independent } N(0, \sigma^2).$$