MCGILL UNIVERSITY
FACULTY OF SCIENCE

PRACTICE FINAL EXAM

MATH 150

Examiner: Professor V. Jaksic

Time: 180 min

Family Name (Please Print): ______________________

First Name: ____________

Student Number: __________

Please do not tear any pages from the exam.

There are eight empty pages at the end of the exam.
You may use them for rough work.

There are 10 questions worth a total of 280 points.

Please provide justification for all your answers.

You will receive no marks for answers without justification.

Write your solutions in a clear, complete and logical way.

1. Fill in the blank with the correct result. To receive marks, you must provide in the space below a correct justification of your answer.

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- [10 points] The equation of the tangent line to the curve

$$\cosh y = x + \sin y + \cos y$$

at the point $(0, 0)$ is

__________________________.

- [10 points] Find a, b, and c such that the parabola $y = ax^2 + bx + c$ passes through the point $(1, 4)$ and its tangent lines at $x = -1$ and $x = 5$ have slopes 6 and -2, respectively.

$a =$ __________ $b =$ __________ $c =$ __________

- [10 points] Suppose that $f(g(x)) = x$ and $f'(x) = 1 + [f(x)]^2$. Then

$g'(x) =$ __________________________

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2. Fill in the blank with the correct result. To receive marks, you must provide in the space below a correct justification of your answer.

- [10 points] The value of the limit

$$\lim_{x\to 0} \frac{\sin((\sin x)^{2009})}{x^{2009}}$$

is

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- [10 points] The inflection points of $f(x) = e^{-x}\sin x$ are

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- [10 points] For which positive numbers a does the curve $y = a^x$ intersects the line $y = x$?

$a =$ ______________

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3. Fill in the blank with the correct result. To receive marks, you must provide in the space below a correct justification of your answer.

- [10 points] The number of positive solutions of the equation $e^x = 1 + x + \frac{x^2}{2}$ is

  ____

- [10 points] For what values of c does the polynomial $P(x) = x^4 + cx^3 + x^2$ have two inflection points?

  $c =$ ____________________________________

- [10 points] Find the area of the largest rectangle that can be inscribed in a right triangle with legs of length 3 cm and 4 cm if two sides of the rectangle lie along its legs.

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4. For what values of a and b is the following equation true?

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$$\lim_{x \to 0} \left(\frac{\sinh(5x)}{x^3} + a - \frac{b}{3x^2} \right) = 0.$$

Justify carefully your answer.

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5. Fill in the blank with the correct result. To receive marks, you must provide in the space below a correct justification of your answer.

- [10 points] The equation of the tangent plane to the surface

$$x - z = 4\arctan(yz)$$

at the point $(1 + \pi, 1, 1)$ is

$$\underline{\hspace{8cm}}.$$

- [10 points] At what point on the paraboloid $y = x^2 + z^2$ is the tangent plane parallel to the plane $x + 2y + 3z = 1$?

$$\underline{\hspace{6cm}}$$

- [10 points] Suppose that a function $F(x, y, z) = 0$ implicitly defines each of the three variables x, y, z as functions of the other two: $x = x(y, z)$, $y = y(x, z)$, $z = z(x, y)$. If F is differentiable and F_x, F_y, F_z are all non-zero, then

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = \underline{\hspace{5cm}}$$

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6. Fill in the blank with the correct result. To receive marks, you must provide in the space below a correct justification of your answer.

- [10 points] The radius of a right circular cone is increasing at a rate of 2 cm/sec while its height is decreasing at a rate of 1 cm/sec. At what rate is the volume of the cone changing when the radius is 10 cm and the height is 10 cm?

  ________________________________.

- [10 points] Let $f(x, y) = x^5y^5$, $x(s, t) = s^2t + s$, $y(s, t) = t + s^2$, $z(s, t) = f(x(s, t), y(s, t))$.

  Using the chain rule, find the value of

  $$\frac{\partial z}{\partial s}(1, 1) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

- [10 points] If $z = x + y + f(x - y) + f(y - x)$ where f is differentiable and $f(0) = 2009$, then

  $$\frac{\partial z}{\partial x}(2009, 2009) + \frac{\partial z}{\partial y}(2009, 2009) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

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7. Fill in the blank with the correct result. To receive marks, you must provide in the space below a correct justification of your answer.

- [10 points] Suppose that a function $f(x, y)$ has, at the point $(1, 2)$, directional derivative 2 in the direction toward $(2, 2)$ and -2 in the direction toward $(1, 1)$. Then

  $$\nabla f(1, 2) = \underline{\hspace{8cm}}.$$

- [10 points] Find the points (x, y) and directions for which the directional derivative of $f(x, y) = 3x^2 + y^2$ has its largest value, if (x, y) is restricted to be on the circle $x^2 + y^2 = 1$.

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- [10 points] A parametric equation of the normal line to the surface

  $$x - z = 4 \arctan(yz)$$

  at the point $(1 + \pi, 1, 1)$ is

  $$x = \_\_\_\_ + t\_\_\_\_ \qquad y = \_\_\_\_ + t\_\_\_\_ \qquad y = \_\_\_\_ + t\_\_\_\_$$

8. State the **Second Derivative Test** for a function $f(x, y)$ of two variables.

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9. Find the local maxima, local minima, and saddle points of

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$$f(x,y) = (x^2 + 2y^2)e^{-x^2-y^2}.$$

Justify carefully your answer.

Additional page for the Problem 9

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10. The base of an aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of aquarium that minimize the cost of the materials. Justify carefully your answer.