Solution to Practice Problems (Part III)

MAT1348, Summer 2020

1. On a wedding ceremony, in how many ways can we take a picture of three guests with the new couple such that the bride and the bridegroom are not side by side?

Solution. There are $2 \times 4! = 48$ ways to arrange these guests with the new couple such that the new couple is side by side. Hence, there are $5! - 2 \times 4! = 72$ ways to arrange these five people so that the new couple is not side by side.

Another way to solve this question: First arrange three quests in $3! = 6$ ways. Then there are four places for the bride and the bridegroom to choose, and every spot can only fit one person. The bride has 4 choices, and the bridegroom has 3 choices. The total number is $6 \times 4 \times 3 = 72$.

2. Find the number of ways to rearrange the letters in word MISSISSIPPI .

Solution. The word has one M, four S's, four I's, and two P's. The number of ways to re-arrange the letters is $C(11; 1, 4, 4, 2) = \dfrac{11!}{11!4!4!2!} = 34650$.

3. Find the number of ways to re-arrange the letters in word ENGINEERING such that all vowels are side by side, and all consonants are side by side.

Solution. There are 2! ways to arrange vowels and consonants. There are $C(5; 2, 2)$ ways to arrange vowels, and $C(6; 3, 2, 1)$ ways to arrange consonants. The total is

$2! \times (5!/3!2!) \times (6!/(3!2!)) = 172800$.

4. Find the coefficient of the term x^{-1} in expansion of $(x^2 + x^{-3})^{12}$.

Solution. The general term of the expansion is $C(12, n)(x^2)^{12-n}(x^{-3})^n = C(12, n)x^{24-5n}$. $24 - 5n = -1$, $n = 5$. The coefficient is $C(12, 5) = 792$.

5. Find the number of ways to distribute 12 identical balls into 4 distinct boxes such that box 1 has at most 2 balls, box 2 has at most 3 balls and box 3 has at most 4 balls.

Solution. The number of ways to distribute 12 identical balls into four distinct boxes is $C(12 + 4 - 1, 4 - 1) = 455$.

The number of ways such that box 1 has at least 3 balls is $C(9 + 4 - 1, 4 - 1) = 220$.
The number of ways such that box 2 has at least 4 balls is $C(8 + 4 - 1, 4 - 1) = 165$.
The number of ways such that box 1 has at least 5 balls is $C(7 + 4 - 1, 4 - 1) = 120$.

The number of ways such that box 1 has at least 3 balls and box 2 has at least 4 balls is $C(5 + 4 - 1, 4 - 1) = 56$.
The number of ways such that box 1 has at least 3 balls and box 3 has at least 5 balls is $C(4 + 4 - 1, 4 - 1) = 35$.
The number of ways such that box 2 has at least 4 balls and box 2 has at least 5 balls is $C(3 + 4 - 1, 4 - 1) = 20$.

The number of ways such that box 1 has at least 3 balls, box 2 has at least 4 balls, and box 3 has at least 5 balls is $C(0 + 4 - 1, 4 - 1) = 1$.

The number of ways such that box 1 has at most 2 balls, box 2 has at most 3 balls, and box 3 has at most 4 balls is $455 - (220 + 165 + 120) + (56 + 28 + 20) - 1 = 53$.

6. Find the number of ways to arrange three couples in a row such that no couple is side by side.

Solution. Define properties:

P_i: Couple i is side by side, $i = 1, 2, 3$.

Let p_i be the number of arrangements that satisfy P_i, $i = 1, 2, 3$.

Let p_{ij} be the number of arrangements that satisfy P_i and P_j, $1 \le i < j \le 3$.

Let p_{123} be the number of arrangements that satisfy all conditions P_1, P_2, and P_3.

Then $p_i = 5!2! = 240$, $i = 1, 2, 3$.

$p_{ij} = 4!2!2! = 96$, $1 \le i < j \le 3$.

$p_{123} = 3!2!2!2! = 48$.

The total number of arrangements of 6 people is $6! = 720$.

By the inclusion-exclusion principle, the number of arrangements that do not satisfy all three conditions is

$720 - 3 \times 240 + 3 \times 96 - 48 = 240$.

7. How many bit string are there with five 1's and seven 0's such that at most three 1's may be side by side.

Solution. We have $C(8, 1)$ bit strings where all five 1's are side by side.

We have $C(8, 2)2!$ bit strings where exactly four 1's are side by side.

Hence, we have $C(12, 5) - C(8, 1) - C(8, 2)2! = 728$ ways where at most three 1's are side by side.

Another way: We have $C(8, 5) = 56$ bit strings no two 1's are side by side.

We have $C(8, 4) \times 4 + C(8, 3) \times 3 = 448$ bit strings with two 1's side by side.

We have $C(8, 3) \times 3 + C(8, 2) \times 2 = 224$ bit strings with three 1's side by side.

$56 + 448 + 224 = 728$.

8. Find the number of bit strings of length 10 such that it has exactly four or five 1's or the first and the last bits are both 0.

Solution. The number of bit strings of length 10 with exactly four 1's is $C(10, 4) = 210$.

The number of bit strings of length 10 with exactly five 1's is $C(10, 5) = 252$.

The number of bit strings of length 10 starting and ending with 0's is $2^8 = 256$.

The number of bit strings of length 10 with exactly four 1's, starting and ending with 0's is $C(8, 4) = 70$.

The number of bit strings of length 10 with exactly five 1's, starting and ending with 0's is $C(8, 5) = 56$.

The total number of bit strings of length 10 with exactly four or five 1's of starting and ending with 0's is $(210 + 252 + 256) - (70 + 56) = 592$.

9. There are five restaurants near my home, and I am going to a restaurant to have dinner every weekday. In how many ways can I going to these restaurants in two weeks (10 weekdays) such that every restaurant is visited at least once?

Solution. Let the restaurants are R_1, R_2, R_3, R_4 and R_5.

If R_i is not visited, $i = 1, 2, 3, 4, 5$, the number of ways is 4^{10}.

If R_i and R_j, $1 \le i < j \le 5$, are not visited, the number of ways is 3^{10}.

If R_i, R_j and R_k, $1 \le i < j < k \le 5$, are not visited, the number of ways is 2^{10}.

If R_i, R_j, R_k and R_m, $1 \le i < j < k < m \le 5$, are not visited, the number of ways is 1^{10}.

If every restaurant is visited at least once, then the number of ways is

$$5^{10} - C(5, 1) \times 4^{10} + C(5, 2) \times 3^{10} - C(5, 3) \times 2^{10} + C(5, 4) \times 1^{10} = 5103000.$$

10. Prove that, if a set S of eight integers are selected randomly, then there must be x and y in S such that $x \equiv y \pmod 7$.

Proof. The remainder of an integer divided by 7 can only take 7 values 0, 1, 2, …, 6. There must be at least two integers x and y in S have the same remainder when divided by 7. Then $x \equiv y \pmod 7$.

11. How many integers do we have to select so that there must exist at least seven selected integers that are congruent modulo 7?

Solution. Since there are 7 possible remainders when divide an integer by 7, we have 7 "holes". We want to ensure that at least 7 integers in the same hole. Hence, we need $\lceil N / 7 \rceil = 7$. The largest number with $\lceil N / 7 \rceil = 6$ is $N = 42$. Then $N = 43$ is the smallest number to have $\lceil N / 7 \rceil = 7$.