Solution to Practice Problems (Part II)

MAT1348, Summer 2020

1. Prove that $\sqrt[3]{3}$ is an irrational number.

Solution. Assume, to the contrary, that $\sqrt[3]{3}$ is a rational number, i.e., $\sqrt[3]{3} = \frac{m}{n}$, where m and n are positive integers, and, when cancel all common factors in the fraction, we may assume that $\frac{m}{n}$ is irreducible.

Now $\frac{m^3}{n^3} = 3$. $m^3 = 3n^3$. Hence, m is a multiple of 3. Let $m = 3k$, where k is an integer. Then $(3k)^3 = 3^3k^3 = 3n^3$, or $n^3 = 9k^2$. Then n is a multiple of 3. Since we assumed that m and n do not have a common factor, this is a contradiction. The result is proved by contradiction.

2. Prove the following result by mathematical induction: When n is an odd positive integer, then $n^2 - 1$ is a multiple of 8.

Solution. Base case: When $n = 1$, $n^2 - 1 = 0$ is a multiple of 8.

Induction hypothesis: Assume that $n^2 - 1$ is a multiple of 8 for an odd integer n.

We want to prove that $(n + 2)^2 - 1$ is a multiple of 8. (Here we use $n + 2$ because $n + 2$ is the next odd integer).

$(n + 2)^2 - 1 = n^2 + 4n + 4 - 1 = (n^2 - 1) + 4(n + 1)$.

By the induction hypothesis, $n^2 - 1$ is a multiple of 8. Since n is odd, $n + 1$ is even. Hence, $4(n + 1)$ is a multiple of 8. Consequently, $(n + 2)^2 - 1 = (n^2 - 1) + 4(n + 1)$ is a multiple of 8.

3. Explain why $\varnothing \subseteq \{\{\varnothing\}\}$, but $\varnothing \notin \{\{\varnothing\}\}$.

Answer. A is a subset of B means that, if $x \in A$, then $x \in B$. If $A = \varnothing$, then the statement $x \in A$ is false because $A = \varnothing$ does not have any element. Since the condition $x \in A$ is false, the statement "$x \in A$ implies $x \in B$" is true. Therefore, the empty set $\varnothing$ is a subset of any set including the set $\{\{\varnothing\}\}$.

The relation $\varnothing \in \{\{\varnothing\}\}$ means that $\varnothing$ is an element of $\{\{\varnothing\}\}$. However, $\{\{\varnothing\}\}$ has only one element that is the set $\{\varnothing\}$, which is not the empty set because it has an element $\varnothing$. Hence, $\varnothing$ is not an element of $\{\{\varnothing\}\}$. The relation $\varnothing \in \{\{\varnothing\}\}$ is false.

4. Let A, B, and C, be sets. Prove the identity $(B - A) \cup (C - A) = (B \cup C) - A$ by the following two methods:

(i) By the definition of the equality of sets.

(ii) By a membership table.

Solution. (i) Let $x \in (B - A) \cup (C - A)$. Then $x \in B - A$ or $x \in C - A$. If $x \in B - A$, then $x \in B$ and $x \notin A$. Hence, $x \in (B \cup C) - A$. If $x \in C - A$, then $x \in C$ and $x \notin A$. We also have, $x \in (B \cup C) - A$. This gives $(B - A) \cup (C - A) \subseteq (B \cup C) - A$.

Conversely, let $x \in (B \cup C) - A$. This means that $x \in B \cup C$ and $x \notin A$. If $x \in B$, then $x \in B - A$. If $x \in C$, then $x \in C - A$. Therefore, $x \in (B - A) \cup (C - A)$.
This gives $(B \cup C) - A \subseteq (B - A) \cup (C - A)$.

By the definition of the equality of two sets, we have $(B - A) \cup (C - A) = (B \cup C) - A$.

(ii) The membership table is the following:

A	B	C	$B - A$	$C - A$	$(B - A) \cup (C - A)$	$B \cup C$	$(B \cup C) - A$
0	0	0	0	0	0	0	0
0	0	1	0	1	1	1	1
0	1	0	1	0	1	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0
1	0	1	0	0	0	1	0
1	1	0	0	0	0	1	0
1	1	1	0	0	0	1	0

Since the column $(B - A) \cup (C - A)$ and the column $(B \cup C) - A$ are identical, we have the identity $(B - A) \cup (C - A) = (B \cup C) - A$.

5. Let $S = \{(x, y) \mid x \text{ and } y \text{ are positive integers}\}$. A relation R from S to S is defined as $(x_1, y_1)\ R\ (x_2, y_2)$ if and only if $x_1 + y_2 = x_2 + y_1$. Show that R is an equivalence relation.

Solution. R is reflexive: If $x_1 = x_2$ and $y_1 = y_2$, then $x_1 + y_2 = x_2 + y_1$. Hence, $(x_1, y_1)\ R\ (x_2, y_2)$. Relation R is reflexive.

R is symmetric: If $(x_1, y_1)\ R\ (x_2, y_2)$, then $x_1 + y_2 = x_2 + y_1$. Hence, $x_2 + y_1 = x_1 + y_2$, i.e., $(x_2, y_2)\ R\ (x_1, y_1)$. Relation R is symmetric.

R is transitive: If $(x_1, y_1)\ R\ (x_2, y_2)$ and $(x_2, y_2)\ R\ (x_3, y_3)$, then $x_1 + y_2 = x_2 + y_1$ and $x_2 + y_3 = x_3 + y_2$. Adding up these equalities, $x_1 + y_2 + x_2 + y_3 = x_2 + y_1 + x_3 + y_2$. Subtract $x_2 + y_2$ on both sides of this equality. We have $x_1 + y_3 = x_3 + y_1$, which gives $(x_1, y_1)\ R\ (x_3, y_3)$. Relation R is transitive.

Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

6. Let $A = \{1, 2, 3, \ldots\}$ be the set of all positive integers, and let $B = \{2, 4, 6, \ldots\}$ be the set of all positive even integers.

(i) Define a function from A to B that is injective but not surjective.

(ii) Define a function from B to A that is surjective but no injective.

(iii) Define a function from A to B that is neither surjective, nor injective.

Justify that the function that you define has the required property.

Solution. The solution is not unique. The following is one of the possible answers:

(i) $f(x) = 2(x + 1)$.

This gives $f(1) = 4$, $f(2) = 6$, $f(3) = 8$, … .

Since 2 is not in the range. This function is not surjective. If $2(x_1 + 1) = 2(x_2 + 1)$, then $x_1 + 1 = x_2 + 1$, and $x_1 = x_2$. This function is injective.

(ii) Let $f(2) = f(4) = 1$, $f(2m) = m - 1$, $m = 3, 4, \ldots$.

This gives $f(6) = 2$, $f(8) = 3$, $f(10) = 4$, … .

Since $f(2) = f(4)$, this function is not injective. For every even integer n in A, since $f(2n + 2) = n$, this function is surjective.

(iii) Let $f(1) = f(2) = 4$. $f(n) = 2n$, when $n \geq 3$. This function is not injective because

$f(1) = f(2)$. It is not surjective because 2 is not in the range.

7. Let f_1 be a function from A to B, and let f_2 be a function from B to C. Prove, by contradiction that, if the composite function $f_2 \circ f_1$ is an injection and f_1 is a surjection, then f_2 is an injection.

Prove. Assume that f_2 is not an injection, then we have $z = f_2(y_1) = f_2(y_2) \in C$, where $y_1, y_2 \in B$, and $y_1 \neq y_2$. Because f_1 is a surjection, we have $y_1 = f_1(x_1)$ and $y_2 = f_1(x_2)$, where $x_1, x_2 \in A$. Since $y_1 \neq y_2$, $x_1 \neq x_2$. Then $(f_2 \circ f_1)(x_1) = f_2(f_1(x_1)) = f_2(y_1) = z$, and $(f_2 \circ f_1)(x_2) = f_2(f_1(x_2)) = f_2(y_2) = z$. The composition $f_2 \circ f_1$ is not an injection. This contradiction proves that f_1 must be an injection.

8. Write $(1101101)_2$ in decimal, octal, and hexadecimal, forms.

Solution. $(1101101)_2 = 1 + 2^2 + 2^3 + 2^5 + 2^6 = 1 + 4 + 8 + 32 + 64 = 109$.

$(1\ 101\ 101)_2 = (155)_8$.

$(110\ 1101)_2 = (6D)_{16}$.

9. Find the greatest common divisor and the least common multiplier of 1869 and 2331.

Solution. Use Euclidean Algorithm.

$2331 = 1 \times 1869 + 462$,
$1869 = 4 \times 462 + 21$,
$462 = 22 \times 21$.

gcd(1869, 2331) = 21.

$$\text{lcm}(1869, 2331) = \frac{1869 \times 2331}{21} = 207459.$$

10. Find the smallest positive integer that satisfies the system of congruences

$x \equiv 7 \pmod{11}$, (1)
$x \equiv 2 \pmod{17}$, (2)
$x \equiv 3 \pmod{23}$. (3)

Solution. From (1), $x = 11r + 7$. Substitute into (2), $11r + 7 \equiv 2 \pmod{17}$,

$11r \equiv -5 \equiv 12 \pmod{17}$. To solve this find the inverse of 11 modulo 17.

$17 = 1 \times 11 + 6$,
$11 = 1 \times 6 + 5$,
$6 = 1 \times 5 + 1$.

Hence, $1 = 6 - 5 = 6 - (11 - 6) = 2 \times 6 - 11 = 2 \times (17 - 11) - 11 = 2 \times 17 - 3 \times 11$.

The inverse of 11 mod 17 is $-3 \equiv 14$.

The solution of congruence $11r \equiv 12 \pmod{17}$ is $r \equiv 12 \times 14 \equiv 168 \equiv 15 \pmod{17}$, or $r = 17s + 15$. Then $x = 11r + 7 = 11(17s + 15) + 7 = 187s + 172$.

Substitute into (3): $x \equiv 187s + 172 \equiv 3 \pmod{23}$, or $187s \equiv -169 \equiv 15 \pmod{23}$.

Find the inverse of 187 modulo 23:

$187 = 8 \times 23 + 3$,
$23 = 7 \times 3 + 2$,
$3 = 2 + 1$.

$1 = 3 - 2 = 3 - (23 - 7 \times 3) = 8 \times 3 - 23 = 8 \times (187 - 8 \times 23) - 23 = 8 \times 187 - 65 \times 23$.

The inverse of 187 modulo 23 is 8.

The solution to congruence $187s \equiv 15 \pmod{23}$ is $s \equiv 15 \times 8 \equiv 120 \pmod{23} = 5 \pmod{23}$.
Then $s = 23t + 5$, and $x = 187s + 172 = 187(23t + 5) + 172 = 4301t + 1107$.

The smallest positive integer solution to this system is $x = 1107$.