Solution to Practice Problems (Part I)

MAT1348, Summer 2020

1. Use the formula $A \to B \equiv {\sim}A \vee B$ to show that $A \to B \equiv {\sim}B \to {\sim}A$.

Solution. $A \to B \equiv {\sim}A \vee B \equiv B \vee {\sim}A \equiv {\sim}{\sim}B \vee {\sim}A \equiv {\sim}B \to {\sim}A$.

2. Construct the truth table of the formula $(A \vee C \equiv B) \to A \wedge {\sim}B$.

Solution.

A	B	C	$A \wedge {\sim}B$	$A \vee C$	$A \vee C \equiv B$	$(A \vee C \equiv B) \to A \wedge {\sim}B$
T	T	T	F	T	T	F
T	T	F	F	T	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	T	F
F	T	F	F	F	F	T
F	F	T	F	T	F	T
F	F	F	F	F	T	F

3. Use a truth table to prove the formula $(A \to B) \wedge (B \to A) \equiv (A \equiv B)$ is a tautology.

Solution.

A	B	C	$A \to B$	$B \to A$	$(A \to B) \wedge (B \to A)$	$A \equiv B$	$(A \to B) \wedge (B \to A) \equiv (A \equiv B)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	F	F	T
T	F	F	F	T	F	F	T
F	T	T	T	F	F	F	T
F	T	F	T	F	F	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Since the truth value in the last column is always true, this formula is a tautology.

4. Find a formula X in CNF, which involves atomic propositions A, B and C with the following truth table (Do not simplify):

A	*B*	*C*	*X*
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	T

Solution. Construct a disjunction clause for each evaluation with X being false

A	*B*	*C*	*X*	disjunction clause
T	T	F	F	$\sim A \vee \sim B \vee C$
T	F	F	F	$\sim A \vee B \vee C$
F	T	T	F	$A \vee \sim B \vee \sim C$

Construct a CNF with these disjunction clauses:

$$X \equiv (\sim A \vee \sim B \vee C) \wedge (\sim A \vee B \vee C) \wedge (A \vee \sim B \vee \sim C).$$

If the question asks to find X in DNF, then X contains five disjunction clauses using the evaluations when X is true:

$$X \equiv (A \vee B \vee C) \wedge (A \vee \sim B \vee C) \wedge (\sim A \vee B \vee \sim C) \wedge (\sim A \vee \sim B \vee C) \wedge (\sim A \vee \sim B \vee \sim C).$$

5. Find a DNF that is equivalent to formula $(A \vee C \equiv B) \rightarrow (A \wedge \sim B)$. Simplify if possible.

Solution. By the properties of connectives, the following formulas are equivalent:

$$(A \vee C \equiv B) \rightarrow A \wedge \sim B$$
$$\equiv \sim(A \vee C \equiv B) \vee (A \wedge \sim B)$$
$$\equiv \sim(((A \vee C) \wedge B) \vee (\sim(A \vee C) \wedge \sim B)) \quad \vee \quad (A \wedge \sim B)$$
$$\equiv (\ \sim((A \vee C) \wedge B) \ \wedge \ \sim(\sim(A \vee C) \wedge \sim B) \ ) \ \vee \ (A \wedge \sim B)$$
$$\equiv (\ (\sim(A \vee C) \vee \sim B) \ \wedge \ ((A \vee C) \vee B) \ ) \ \vee \ (A \wedge \sim B)$$
$$\equiv (\sim(A \vee C) \wedge (A \vee C)) \ \vee \ (\sim(A \vee C) \wedge B) \ \vee \ (\sim B \wedge (A \vee C)) \ \vee \ (\sim B \wedge B)) \ \vee \ (A \wedge \sim B)$$
$$\equiv (\sim(A \vee C) \wedge B) \ \vee \ (\sim B \wedge (A \vee C)) \ \vee \ (A \wedge \sim B)$$
$$\equiv (\sim A \wedge \sim C \wedge B) \ \vee \ (\sim B \wedge (A \vee C)) \ \vee \ (A \wedge \sim B)$$
$$\equiv (\sim A \wedge \sim C \wedge \sim B) \ \vee \ (\sim B \wedge A) \ \vee \ (\sim B \vee C) \ \vee \ (A \wedge \sim B)$$

$\equiv (\sim A \wedge B \wedge \sim C) \vee (A \wedge \sim B) \vee (\sim B \wedge C)$.

Another way is to construct the truth table of this formula. Then construct a DNF using evaluations when this formula is true.

A	B	C	$A \wedge \sim B$	$A \vee C$	$A \vee C \equiv B$	$(A \vee C \equiv B) \rightarrow (A \wedge \sim B)$
T	T	T	F	T	T	F
T	T	F	F	T	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	T	F
F	T	F	F	F	F	T
F	F	T	F	T	F	T
F	F	F	F	F	T	F

The DNF is

$(A \wedge \sim B \wedge C) \vee (A \wedge \sim B \wedge \sim C) \vee (\sim A \wedge B \wedge \sim C) \vee (\sim A \wedge \sim B \wedge C)$,

which is equivalent to the previous solution.

Indeed, expend to complete DNF:

$(\sim A \wedge B \wedge \sim C) \vee (A \wedge \sim B) \vee (\sim B \wedge C)$
$\equiv (\sim A \wedge B \wedge \sim C) \vee (A \wedge \sim B \wedge C) \vee (A \wedge \sim B \wedge \sim C) \vee (A \wedge \sim B \wedge C) \vee (\sim A \wedge \sim B \wedge C)$
$\equiv (\sim A \wedge B \wedge \sim C) \vee (A \wedge \sim B \wedge C) \vee (A \wedge \sim B \wedge \sim C) \vee (\sim A \wedge \sim B \wedge C)$.

6. Define atomic propositions as follows:

A: *A* is a knight.
B: *B* is a knight.
C: *C* is a knight.

Then

~*A*: *A* is a knave.
~*B*: *B* is a knave.
~*C*: *C* is a knave.

Translate each of the following statements into a propositional logic formula using the atomic propositions defined above:

a. *A* is a knight if and only if *B* and *C* are knaves.

b. *A* is a knight only if *B* or *C* is a knave.

c. "*A* is a knight" is a necessary condition of that *B* is a knave.

d. Among *A*, *B*, and *C*, exactly two of them is a knight.

e. Among *A*, *B*, and *C*, at most two of them are knaves.

f. *A* is a knight whenever exactly one of *B* and *C* is a knight.

g. Even though *A* is a knight, *B* or *C* is a knave.

h. Even if both *A* and *B* are knaves, *C* does not have to be a knave.

Solution.

a. $A \equiv {\sim}B \wedge {\sim}C$.

b. ${\sim}({\sim}B \vee {\sim}C) \rightarrow A$ or $A \rightarrow {\sim}B \vee {\sim}C$.

c. ${\sim}B \rightarrow A$.

d. $({\sim}A \wedge B \wedge C) \vee (A \wedge {\sim}B \wedge C) \vee (A \wedge B \wedge {\sim}C)$

e. $A \vee B \vee C$ or ${\sim}({\sim}A \wedge {\sim}B \wedge {\sim}C)$.

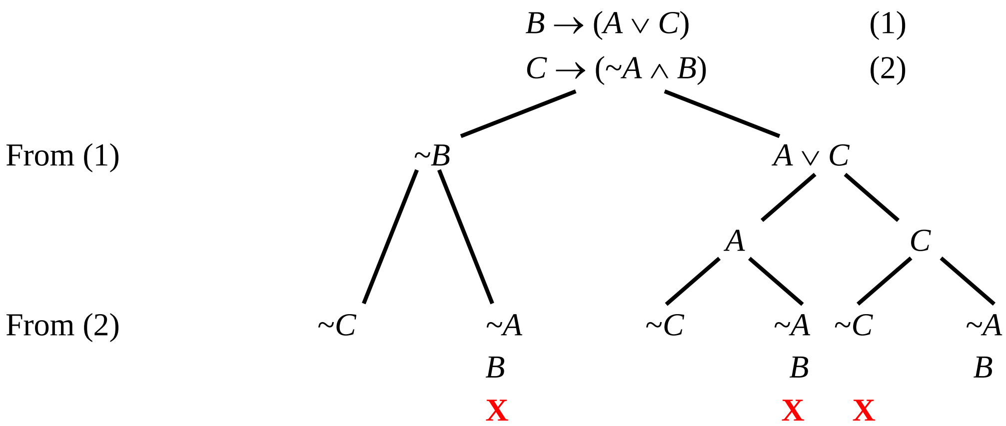
f. $(B \wedge {\sim}C) \vee ({\sim}B \wedge C) \rightarrow A$.

g. $A \wedge ({\sim}B \vee {\sim}C)$.

h. Restate the sentence: It is not true that, if *A* and *B* are knaves, then *C* must be a knave. The formula is ${\sim}(({\sim}A \wedge {\sim}B) \rightarrow {\sim}C)$.

7. Find a formula *P* that involves only *B* and *C* such that the set of formulas $\{ B \rightarrow (A \vee C), C \rightarrow ({\sim}A \wedge B), P\}$ is inconsistent.

Solution. Construct a truth tree using the first two formulas:



The open branches give the following evaluations so that the first two formulas are true:

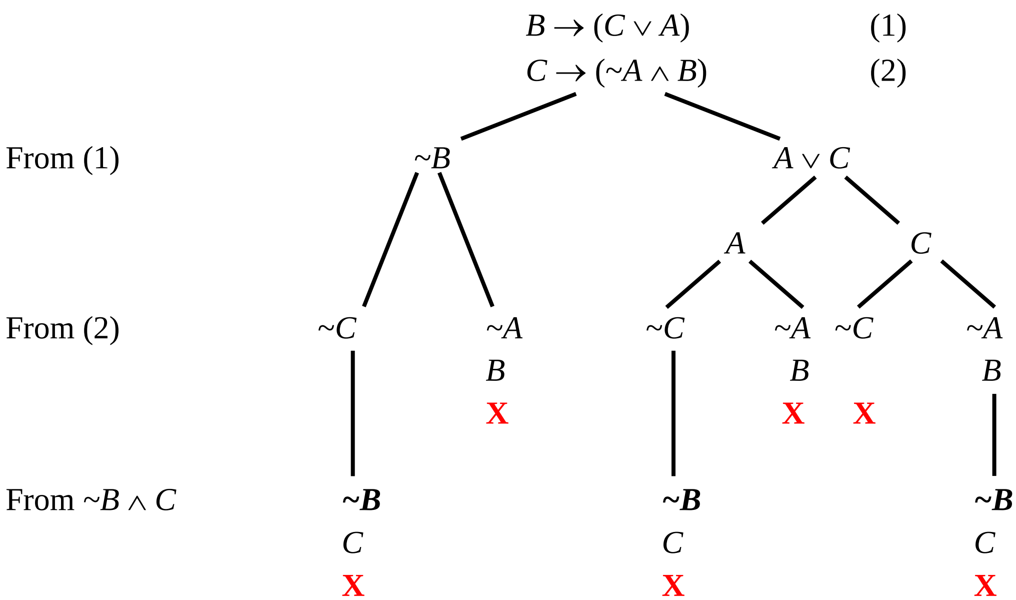
A	*B*	*C*
F	F	F
T	F	F
T	T	F
F	T	T

Hence, P must satisfy the condition that P is false for any of these four evaluations. We need a formula with B and C that is false for all these four evaluations. The simplest one is $P \equiv \sim B \wedge C$, which is true only if B is false, and C is true.

The truth tree of $\sim B \wedge C$ is

$\sim B$
C

Add this to the end of open branches. We see that all branches are closed.



8. On the island of knights and knaves, is the following argument valid? If it is not, give a counter-example, i.e., an evaluation such that the premises are true, but the conclusion is false.

Premises:

A is a knight if and only if *B* is a knave and *C* are knights.
If *C* and *D* are both knights, then *E* is a knight.
E and *B* are knaves.

Conclusion:

A is a knave.

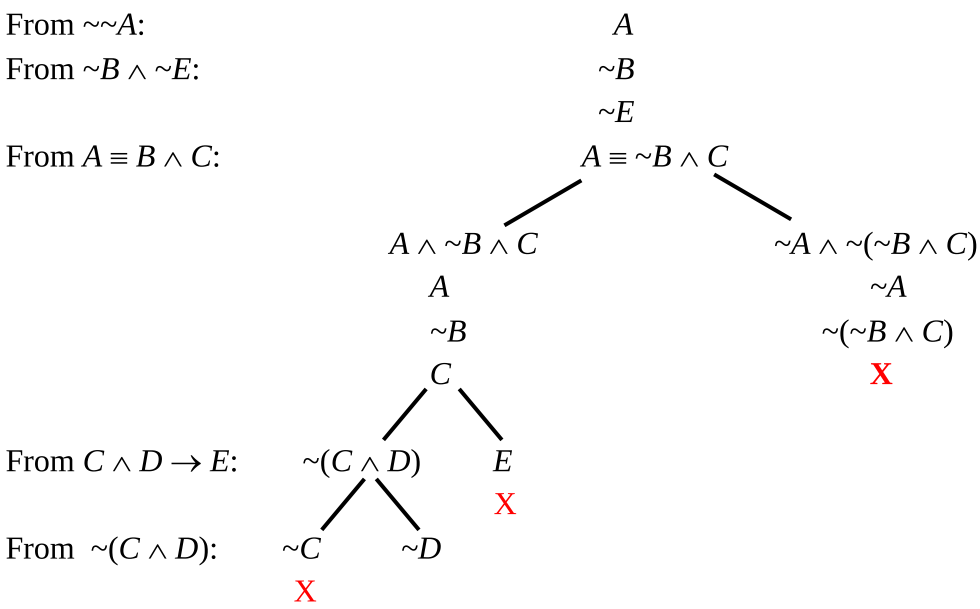
Solution. Define propositions: *X* to mean that *X* is a knight, where *X* stands for *A*, *B*, *C*, *D*, or *E*.

Then the argument can be represented by $\{A \equiv {\sim}B \wedge C,\ C \wedge D \rightarrow E,\ {\sim}B \wedge {\sim}E \mid {\sim}A\}$.

To determine whether this argument is valid, use a truth tree to check if the set of formulas

$\{A \equiv {\sim}B \wedge C,\ C \wedge D \rightarrow E,\ {\sim}B \wedge {\sim}E,\ {\sim}{\sim}A\}$

is inconsistent.



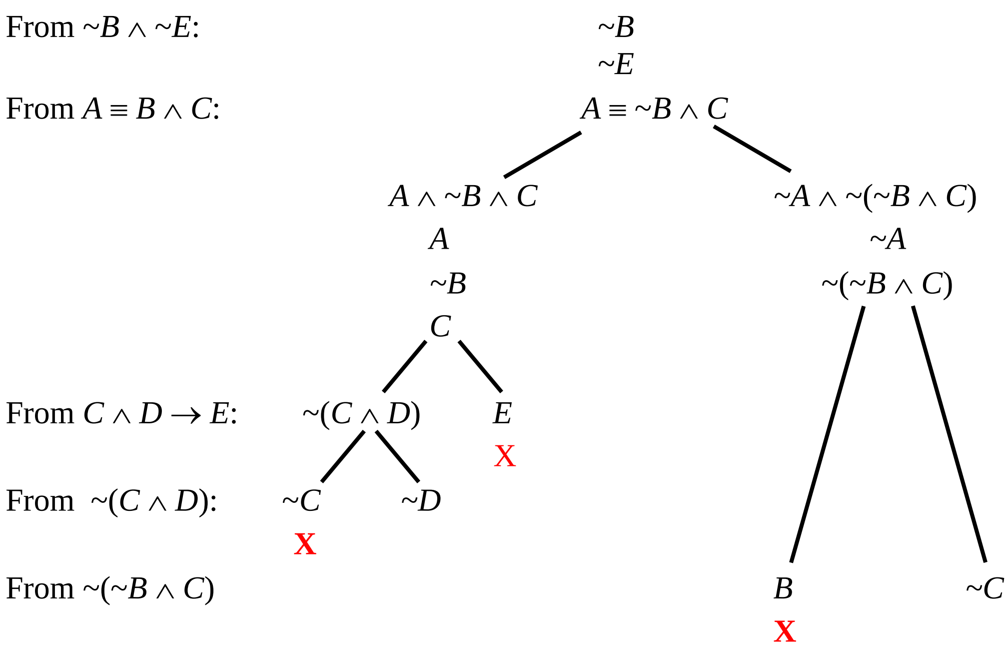
There is one open branch. The argument is invalid. The open branch give an evaluation: *A*: true, *B*: false, *C*: true, *D*: false, and *E*: false.

In other words, if *A* and *C* are knights, and *B*, *D*, and *E*, are knaves, then the premises are true, and the conclusion is false.

9. Which one of the following formulas can be used to replace the conclusion (i.e., $\sim A$) in the previous argument so that the argument is valid?

(1) $A \vee C$; (2) $A \vee \sim C$; (3) $A \vee D$; (4) $A \vee \sim D$; (5) $C \vee D$; (6) $\sim C \vee D$.

Solution. Construct the truth tree of the premises: $\{A \equiv \sim B \wedge C,\ C \wedge D \rightarrow E,\ \sim B \wedge \sim E\}$



The open branches give the following evaluations so that the premises are all true:

	A	B	C	D	E
(i)	T	F	T	F	F
(ii)	F	F	F	T	F
(iii)	F	F	F	F	F

The conclusion should be true for all these evaluations.

Only (2) $A \vee \sim C$ satisfies this condition.

(1) is false for evaluation (i), (3) is false for evaluation (iii), (4) is false for evaluation (ii), (5) is false for evaluation (iii), (6) is false for evaluation (i).

10. Define the following predicates:

Px: x is a person in this party.
Sy: y is a spicy dish.
Exy: x eats y.

Translate each of the following sentences into a predicate logic formula using the predicates defined above:

a. Everybody in this party eats a certain spicy dish.

b. Someone in this party eats all spicy dishes.

c. Someone in this party does not eat any spicy dish.

d. One spicy dish is eaten by everybody in this party.

e. Every one in this party eats every spicy dish.

Solution. a. $\forall x\,(Px \to \exists y\,(Sy \wedge Exy))$.

b. $\exists x\,(Px \wedge \forall y\,(Sy \to Exy))$.

c. $\exists x(Px \wedge \forall y\,(Sy \to \sim Exy))$.

d. $\exists y\,(Sy \wedge \forall x\,(Px \to Exy))$.

e. $\forall x\,(Px \to \forall y\,(Sy \to Exy))$.

11. Write the negation of the predicate logic formula (a) and (b) that you created in the previous question such that the negation is only before a predicate, and state the sentence that this formula represents.

a. $\sim(\forall x\,(Px \to \exists y\,(Sy \wedge Exy))) \equiv \exists x(\sim(\sim Px \vee \exists y\,(Sy \wedge Exy))) \equiv \exists x(Px \wedge \sim(\exists y\,(Sy \wedge Exy)))$
$\equiv \exists x(Px \wedge \forall y(\sim(Sy \wedge Exy))) \equiv \exists x(Px \wedge \forall y(\sim Sy \vee \sim Exy)) \equiv \exists x(Px \wedge \forall y(Sy \to \sim Exy))$.

There is a person in this party who does not eat any spicy dish.

b. $\sim(\exists x\,(Px \wedge \forall y\,(Sy \to Exy))) \equiv \forall x(\sim(Px \wedge \forall y\,(Sy \to Exy))) \equiv \forall x(\sim Px \vee \sim(\forall y\,(\sim Sy \vee Exy)))$
$\equiv \forall x(\sim Px \vee \exists y(\sim(\sim Sy \vee Exy))) \equiv \forall x(\sim Px \vee \exists y(Sy \wedge \sim Exy))) \equiv \forall x(Px \to \exists y(Sy \wedge \sim Exy)))$.

Everyone in this party missed at least one spicy dish.