DGD 6
MAT1348X
June 30, 2020

1. Find the number of ways to re-arrange the letters of word SEERESS.

Solution. This number is to put seven distinct "places" into three distinct "boxes" S, E and R, such that box S has three places, box E has three places and box R has one place. This number is

$$\frac{7!}{3!3!1!} = 140.$$

2. How many different ways can four players in a bridge game to receive their cards with 13 each from a standard deck of 52 cards?

Solution. This question is to distribute 52 different cards to four players such that every player gets 13 cards. This number is $\frac{52!}{13!13!13!13!}$, which is a very big number approximately 5.4×10^{28}.

3. Find the number of ways to distribute 20 identical balls into 4 distinct boxes such that box 1 has at least 3 balls, box 2 has at least one ball and at most 5 balls, and box 3 has at most 4 balls.

Solution. Put three balls into box 1 and one ball into box 2 first. Then we want to distribute 16 balls into these 4 boxes such that box 2 has at most 4 balls, and box 3 has at most 4 balls.

The total number of ways to distribute 16 balls into four boxes is $C(16 + 4 - 1, 4 - 1) = 969$.
The number of ways to distribute 16 balls into four boxes such that box 2 has at least 5 balls is $C(11 + 4 - 1, 4 - 1) = 364$.
The number of ways to distribute 16 balls into four boxes such that box 3 has at least 5 balls is $C(11 + 4 - 1, 4 - 1) = 364$.
The number of ways to distribute 16 balls into four boxes such that box 2 has at least 5 balls **and** box 3 has at least 5 balls is $C(6 + 4 - 1, 4 - 1) = 84$.
The number of ways to distribute 16 balls into four boxes such that box 2 has at least 5 balls **or** box 3 has at least 5 balls is $364 + 364 - 84 = 644$.
The number of ways to distribute 16 balls into four boxes such that box 2 has at most 4 balls **and** box 3 has at most balls is $969 - 644 = 325$.

4. I have 10 pairs of shoes mixed up in a basket. How many shoe I have to take in dark so that there must be a matching pair?

Answer. 11 shoes. When I put 11 shoes into 10 pigeonholes, there are must be two shoes are in the same hole, which is a matching pair.

5. Show that, in a group of people there must be at least two people who have the same number of friends in this group. (Assume that, if x is a friend of y, then y is a friend of x.)

Proof. Suppose this group has n people. Then the number of friends of a person can be one of the numbers 0, 1, …, $n - 1$. However, if one person has $n - 1$ friends, then everyone in this group is a friend of this person, and there does not exist a person with 0 friend. Hence, at least one of the "pigeonholes" is empty. There are always at most $n - 1$ "holes" available. By the pigeonhole principle, there must be two people in the same "hole". In other words, there must be two people in this group who have the same number of friends.

6. Show that, in any set of 12 integers, there must be two different integers x and y such that $x - y$ is a multiple of 11.

Proof. Divide an integer by 11, the remainder can be one of the eleven integers 0, 1, …, 10. In a set of 12 integers, there must be two of them with the same remainder when divided by 11. Then the difference of these two integers is a multiple of 11.

7. What is the smallest number of balls to distribute into 5 boxes such that at least one box has at least 6 balls?

Solution. By the pigeonhole principle, this number is the smallest number N such that $\left\lceil \frac{N}{5} \right\rceil = 6$.

Since $\frac{25}{5} = 5$, the smallest number N that satisfies this equation is 26.

8. How many numbers do we have to take from the set $S = \{1, 3, 5, 7, 9, 11, 13, 15\}$ so that we have at least two numbers with a sum 16?

Solution. We have 4 "boxes": box 1 accepts 1 and 15, box 2 accepts 3 and 13, box 3 accepts 5 and 11, and box 4 accepts 7 and 9. If we take at least 5 numbers from set S, and put the numbers in "their" boxes. Then at least one box has two numbers with a sum 16.

9. Show that if the set of integers {1, 2, …, 16} is separated into three disjoint subsets, then there must exist a subset with two numbers x and y such that $x - y$ is also in this subset.

Solution. This question is a generalization of an example covered in class.

It is proved by contradiction. Assume, to the contrary, there is a way to separate this set into three disjoint subsets S_1, S_2, and S_3, such that if x and y are in a subset, then $x - y$ is not in this subset.

First, by the pigeonhole principle, one of the subsets, say S_1, must have at least 6 numbers: $a_1 < a_2 < \ldots < a_6$. Consider the numbers $b_1 = a_6 - a_1$, $b_2 = a_6 - a_2$, $b_2 = a_6 - a_3$, $b_4 = a_6 - a_4$, $b_5 = a_6 - a_5$. These numbers are not in S_1. Then they are in S_2 or S_3.

By the pigeonhole principle again, there are at least three numbers b_i, b_j, and b_k, $1 \leq i < j < k \leq 5$, in the same subset, say S_2. Now look at $c_j = b_i - b_j$ and $c_k = b_i - b_k$. They are not in S_2. Also, since $b_i - b_j = (a_6 - a_i) - (a_6 - a_j) = a_j - a_i$ and $b_i - b_k = (a_6 - a_i) - (a_6 - a_k) = a_k - a_i$, c_j and c_k are not in S_1. Then they must be in S_3.

Now consider the number $d = c_k - c_j$. By the property of the subsets, d is not in S_3. However, $d = c_k - c_j = (b_i - b_k) - (b_i - b_j) = b_j - b_k = (a_6 - a_j) - (a_6 - a_k) = a_k - a_j$, d is not in S_1 or S_2. Then $d$is not in any of S_1, S_2, or S_3. This is a contradiction.

Question to discuss: Is it possible to separate the set {1, 2, …, 15} into three disjoint subsets such that if x and y are in a subset, then $x - y$ is not in this set?