

DGD 4

MAT1348X

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1. Determine which ones of the following relations are true:

$\emptyset \in \emptyset;$	F
$\emptyset \subseteq \emptyset;$	T
$\emptyset \in \{\emptyset\};$	T
$\emptyset \subseteq \{\emptyset\};$	T
$\emptyset \in \{\{\emptyset\}\};$	F
$\emptyset \subseteq \{\{\emptyset\}\};$	T
$\{\emptyset\} \in \emptyset;$	F
$\{\emptyset\} \subseteq \emptyset;$	F
$\{\emptyset\} \in \{\emptyset\};$	F
$\{\emptyset\} \subseteq \{\emptyset\};$	T
$\{\emptyset\} \in \{\{\emptyset\}\};$	T
$\{\emptyset\} \subseteq \{\{\emptyset\}\}.$	F

2. Let A , B , and C , be three sets. Use a membership table to prove the identity $(A \oplus B) \cap C = (A \cap C) \oplus (B \cap C)$.

Solution.

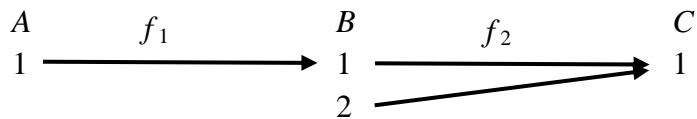
A	B	C	$A \oplus B$	$(A \oplus B) \cap C$	$A \cap C$	$B \cap C$	$(A \cap C) \oplus (B \cap C)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	1	0	1	1
1	0	0	1	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	0	0	1	1	0
1	1	1	0	0	1	1	0

Since the column $(A \oplus B) \cap C$ and the column $(A \cap C) \oplus (B \cap C)$ are identical, the identity $(A \oplus B) \cap C = (A \cap C) \oplus (B \cap C)$ is true.

3. We know that if the composite function $f_2 \circ f_1$ is injective, then f_1 is injective; if $f_2 \circ f_1$ is surjective, then f_2 is surjective. Give counter-examples to show that, if $f_2 \circ f_1$ is injective, f_2 may not be injective, and if $f_2 \circ f_1$ is surjective, f_1 may not be surjective.

Solution. There are many counter-examples. For instance, we can give the following:

Let $A = C = \{1\}$, $B = \{1, 2\}$. $f_1: A \rightarrow B$, such that $f_1(1) = 1$. $f_2: B \rightarrow C$, such that $f_2(1) = f_2(2) = 1$. Then $(f_2 \circ f_1)(1) = f_2(f_1(1)) = f_2(1) = 1$. Hence $f_2 \circ f_1$ is injective and surjective. However, f_1 is not surjective, and f_2 is not injective.



4. Let $S = \{ \emptyset, \{\emptyset\}, \{\{\emptyset\}\} \}$. Find the Cartesian product $S \times S$ and $\text{Pow}(S)$.

Solution. S has three elements. The Cartesian product $S \times S$ has 9 elements, which are ordered pairs of elements of S :

$$\begin{aligned}
 S \times S = \{ & (\emptyset, \emptyset), & (\emptyset, \{\emptyset\}), & (\emptyset, \{\{\emptyset\}\}), \\
 & (\{\emptyset\}, \emptyset), & (\{\emptyset\}, \{\emptyset\}), & (\{\emptyset\}, \{\{\emptyset\}\}), \\
 & (\{\{\emptyset\}\}, \emptyset), & (\{\{\emptyset\}\}, \{\emptyset\}), & (\{\{\emptyset\}\}, \{\{\emptyset\}\}) \\
 & \}.
 \end{aligned}$$

The power set $\text{Pow}(S)$ has eight subsets of S :

$$\begin{aligned}
 \text{Pow}(S) = \{ & \emptyset, & // \text{ The empty set} \\
 & \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, & // \text{ The singletons} \\
 & \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\} & // \text{ The pairs} \\
 & \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\} & // \text{ The triple} \\
 & \}.
 \end{aligned}$$

5. Let \mathbf{R} be the set of real numbers. Define a relation f from \mathbf{R} to \mathbf{R} as follows: If x and y are real numbers, $x f y$ if $x - y$ is an integer. Show that this relation is an equivalence relation.

Solution. Reflexive. Let $x \in \mathbf{R}$. Since $x - x = 0$ is an integer, $x f x$, and relation f is reflexive.

Symmetric. Let $x, y \in \mathbf{R}$. $x f y$ means $x - y$ is an integer. Then $y - x = -(x - y)$ is also an integer. Hence, $y f x$, and relation f is symmetric.

Transitive. Let $x, y, z \in \mathbf{R}$. If xfy and yfz , then $x - y$ is an integer, and $y - z$ is an integer. Since $x - z = (x - y) + (y - z)$ is an integer, xfz , and relation f is transitive.

Because relation f is reflexive, symmetric, and transitive, f is an equivalence relation.

6. Write the following numbers in Octal form:

234567, $(10011010)_2$, $(B1C)_{16}$.

Solution. $234567 = 8 \times 29320 + 7$, $29320 = 8 \times 3665 + 0$, $3665 = 8 \times 458 + 1$, $458 = 8 \times 57 + 2$, $57 = 8 \times 7 + 1$. $234567 = (712107)_8$.

$(10\ 011\ 010)_2 = (232)_8$.

$(B1C)_{16} = (101,1\ 00,01\ 1,100)_2 = (5434)_8$.

7. Find $\gcd(3738, 1554)$.

Solution. $3738 = 2 \times 1554 + 630$,
 $1554 = 2 \times 630 + 294$,
 $630 = 2 \times 294 + 42$,
 $294 = 7 \times 42$.

$\gcd(2728, 1554) = 42$.

8. Write 42 as an integer linear combination of 3738 and 1554.

$42 = 630 - 2 \times 294 = 630 - 2 \times (1554 - 2 \times 630) = (-2) \times 1554 + 5 \times 630$
 $= (-2) \times 1554 + 5 \times (3738 - 2 \times 1554) = 5 \times 3738 - 12 \times 1554$.