DGD 3

MAT1348X
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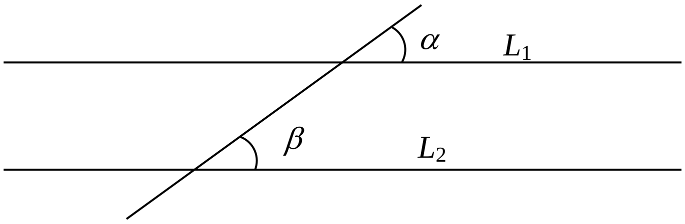
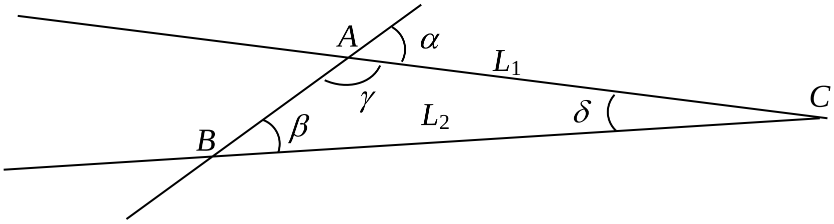
1. Prove: Let x and y be two positive integers. If the geometric mean of x and y, $\sqrt{xy}$ equals the arithmetic mean of x and y, $\frac{x+y}{2}$, are different, then $x \neq y$.

Proof: Prove by contradiction: Assume, to the contrary, that $x = y$, then $\sqrt{xy} = \sqrt{x^2} = x$, and $\frac{x+y}{2} = \frac{2x}{2} = x$. This contradict the assumption that $\sqrt{xy} \neq \frac{x+y}{2}$. Therefore, $x \neq y$.

2. Prove: If two lines L_1 and L_2 are cut by a transversal, and two corresponding angles α and β are equal, as shown in the following figure, then these two lines are parallel.



Proof. Prove b contradiction. Assume, to the contrary, that L_1 and L_2 are not parallel. Then they have an intersection, as in the following figure:



Since $\alpha = \beta$, and $\alpha + \gamma = 2\pi$, we have $\beta + \gamma = 2\pi$. In triangle ABC, the sum of inner angles $\beta + \gamma + \delta = 2\pi + \delta > 2\pi$. this violates the theorem in geometry that the sum of the inner angles of a triangle is 2π radians. This contradiction proves that L_1 and L_2 must be parallel.

3. Prove that $4^{n+1} + 5^{2n-1}$ is a multiple of 21 for any $n \geq 1$, by mathematical induction.

Proof. Prove by the week version of mathematical induction.

Base case: $n = 1$. $4^2 + 5 = 21$ is a multiple of 21.

Induction step:

Induction hypothesis: Assume $4^{k+1} + 5^{2k-1}$ is a multiple of 21 for some $k \geq 1$.

We want to prove that $4^{(k+1)+1} + 5^{2(k+1)-1}$ is a multiple of 21.

Indeed,

$4^{(k+1)+1} + 5^{2(k+1)-1} = 4^{k+2} + 5^{2k+1} = 4 \times 4^{k+1} + 25 \times 5^{2k-1} = (25 - 21) \times 4^{k+1} + 25 \times 5^{2k-1} = 25 \times (4^{k+1} + 5^{2k-1}) - 21 \times 4^{k+1}$.

(Because the induction hypothesis is about $4^{k+1} + 5^{2k-1}$, we must create such a term to be able to use the induction hypothesis! This is why we have write $4^{k+2} = 4 \times 4^{k+1}$ and $5^{2k+1} = 25 \times 5^{2k-1}$. Furthermore, we have to make a multiple of $4^{k+1} + 5^{2k-1}$. This is why we want to write $4 \times 4^{k+1} = (25 - 21) \times 4^{k+1}$.)

By the induction hypothesis, the first term is a multiple of 21. The second term is obviously a multiple of 21. Therefore, $4^{(k+1)+1} + 5^{2(k+1)-1}$ is a multiple of 21.

By the principle of mathematical induction, $4^{n+1} + 5^{2n-1}$ is a multiple of 21 for any $n \geq 1$.

4. Use the strong version of mathematical induction to prove that every integer $n \geq 2$ can be divided by a prime number.

Solution. The base case: When $n = 2$, 2 is divided by itself. Since 2 is a prime number, the conclusion is true when $n = 2$.

Induction hypothesis: Assume that every integer k, $2 \leq k \leq n$, can be divided by a prime number.

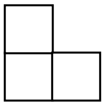
We want to prove that $n + 1$ can also be divided by a prime number.

If $n + 1$ is itself a prime number, then $n + 1$ divides itself.

If $n + 1$ is not a prime number, then $n + 1 = xy$, where x and y are integers, $2 \leq x, y \leq n$. By the induction hypothesis, x is divided by a prime number, $x = pm$, where p is a prime number, and $m \geq 1$. Therefore, $n + 1 = xy = pmy$. In other words, $n + 1$ is divided by a prime number p.

By the mathematical induction principle, every integer $n \geq 2$ can be divided by a prime number.

5. Let B be chessboard with 2^n rows and 2^n columns. Show that, if one square is cut off from the chessboard, the remaining part can be covered by L-shaped dominos as shown in the following figure:



Proof. Prove by the strong version of mathematical induction on n.

The base case: If $n = 0$, the only square is cut off. There is nothing left. The remaining part can be covered by 0 domino.

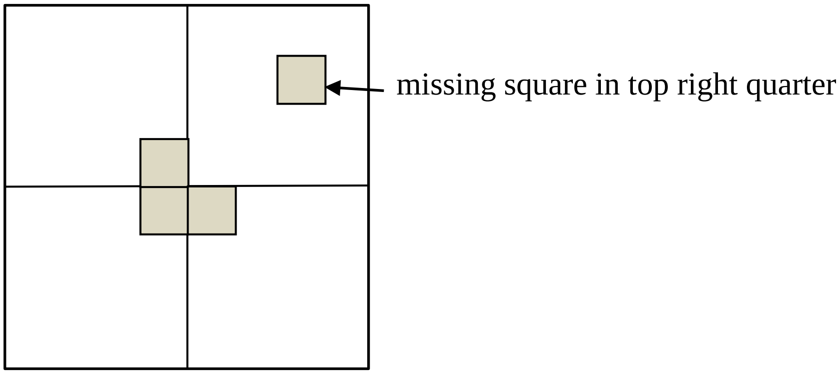
The induction step:

Induction hypothesis: Assume that the conclusion is true for the case $n = k$, $k \geq 0$. I.e., a chessboard with dimension $2^k \times 2^k$, $k \geq 0$, with one square missing can be covered by L-shaped dominos.

We want to show that this proposition is also true for the case $n = k + 1$. I.e., a chessboard with dimension $2^{k+1} \times 2^{k+1}$, $k \geq 0$, with one square missing can be covered by L-shaped dominos.

Assume we have a chessboard with dimension 2^{k+1} by 2^{k+1}. It can be separated into four quarters with dimension 2^k by 2^k by a vertical line and a horizontal line through the center of the chessboard.

Without loss of generality, (if necessary, turn the chessboard around), we may assume that the missing square is in the top right quarter. Now we put a domino to cover one square in each of the three remaining quarters around the center, as the shaded part in the following figure:



Since each of four quarters has a square cut off, by the induction hypothesis, they can be covered by *L*-shaped dominos.

By mathematical induction principle, we know that every chessboard with one square missing can be covered by *L*-shaped dominos.

6. What is wrong with the following "proof"?

"Theorem". If we have a sequence $S = (a_1, a_2, \ldots, a_k)$ of n integers, $n \geq 1$, then all integers in this sequence are equal, i.e., $a_1 = a_2 = \ldots = a_k$.

"Proof". Base case: $n = 1$. Since we have only one integer in this sequence, this integer is, trivially, equal to itself.

Induction hypothesis. Assume all integers in a sequence of k integers are equal.

We want to prove that all integers in a sequence with $k + 1$ integers are equal.

Let $S = (a_1, a_2, \ldots, a_k)$ be a sequence with k integers. By the induction hypothesis, $a_1 = a_2 = \ldots = a_k$. Add one more member $a_{k+1} = a_k$ in this sequence. We have a sequence $S' = (a_1, a_2, \ldots, a_k, a_{k+1})$ with $k + 1$ integers and all members in S' are equal.

The proof is complete.

Answer. You should not start with a sequence with k members, and ADD a new member at the end. In this way, what you show is the following:

If all sequences with k members has the property that all members are equal, then THERE IS a sequence with $k + 1$ members such that all members are equal.

However, the induction principle wants a universal conclusion as follows:

If all sequences with k members have the property that all members are equal, then in ALL sequences with $k + 1$ members, all members are equal.