Applied Ordinary Differential Equations, ENGR-213, Section XX, Fall 2015

2.2. Separable Differential Equations. Method of Separation of Variables to Solve First Order Linear and Non-Linear Differential Equations.

Given a first-order DE
$$F(x, y, y^{'}) = 0.$$
Suppose that the given DE can be represented in a normal form:
$$y^{'} = G(x, y)$$
or by using the Leibnitz notation of derivative we have:
$$\frac{dy}{dx} = G(x, y).$$
If the right-hand side of the above first-order DE can be represented in a form:
$$G(x, y) = f(x)g(y),$$
where $f(x)$ is a function only in the variable x; and $g(y)$ is a function only in the variable y, then the given DE is called **separable**. Hence, separable is a first-order DE assuming the form:
$$\frac{dy}{dx} = f(x)g(y).$$
We solve a separable first-order DE by applying the method of separation of the variables:
(1) We represent the given DE in a normal form by using Leibnitz notation of derivative:
$$\frac{dy}{dx} = G(x, y).$$
(2) If possible, we factor $G(x, y)$ in a form $G(x, y) = f(x)g(y)$. If this is possible the given DE is separable.
(3) We separate the variables, representing the given DE in a differential form such that the function in front of dy depends only on y and the function in front of dx depends only on x:
$$\frac{dy}{dx} = f(x)g(y) \quad \Rightarrow \quad \frac{1}{g(y)}dy = f(x)dx \quad \Rightarrow \quad \frac{dy}{g(y)} = f(x)dx.$$

(4) We integrate both sides of the above differential form:

$$\int \frac{dy}{g(y)} = \int f(x)\, dx,$$

the left-hand side in terms of y and the right-hand side in terms of x, in order to obtain the solution.
(5) The solution is **a one-parameter family** presented either in **an implicit form** or in **an explicit form. Hence, we obtain, either implicit or explicit solution. Our final goal is to obtain an explicit solution, if possible.**

Remark. (A) Note that **we can solve linear and non-linear first-order DEs by using the method of separation of the variables.**
(B) The method requires the given DE to be representable in **a normal form** and the right-hand side of the normal form to be **separable**, i.e., to be representable in the form $f(x)g(y)$.
(C) **If the first derivative is given in a prime notation ($y^{'}$), then the prime notation has to be replaced by the Leibnitz notation (dy/dx) of derivative in order to proceed with the separation of the variables.**

Example. Solve the given first-order DE

$$xy^{'} = 1 + y.$$

Solution. We apply the method of separation of the variables.
Step 1. First we represent the given DE in **a normal form** by using **the Leibnitz notation of derivative**:

$$y^{'} = \frac{1+y}{x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1+y}{x}.$$

Step 2. The DE is separable because the right-hand side permits representation of the form $f(x)g(y)$. We factor the right-hand side in a form $f(x)g(y)$:

$$\frac{1+y}{x} = \frac{1}{x}(1+y) \quad \text{with} \quad f(x) = \frac{1}{x} \quad \text{and} \quad g(y) = 1 + y.$$

Step 3. Next, we separate the variables **representing the given DE in a differential form**:

$$\frac{dy}{1+y} = \frac{dx}{x}.$$

Step 4. Integrating both sides of the above separable differential form of the given DE we obtain:

$$\int \frac{dy}{1+y} = \int \frac{dx}{x} \quad \Rightarrow \quad \ln|1+y| + c_1 = \ln|x| + c_2$$
$$\Rightarrow \quad \ln|1+y| = \ln|x| + c_2 - c_1 \quad \Rightarrow \quad \ln|1+y| = \ln|x| + c$$

replacing the notation $c_2 - c_1$ for an arbitrary constant by c. Hence, in the above formula c is an arbitrary constant.
From here,

$$\ln|1+y| = \ln|x| + c, \quad (c \text{ an arbitrary constant})$$

is **an implicit solution of the given DE**.
Step 5. Our final goal must be to get an explicit solution, if possible. In order **to obtain an explicit solution** we proceed as it follows:

$$e^{\ln|1+y|} = e^{\ln|x|+c} \quad \Rightarrow |1+y| = e^c e^{\ln|x|} \quad \Rightarrow \quad |y+1| = e^c|x|.$$

Replacing e^c by another positive constant $c > 0$ we obtain:

$$|y+1| = c|x|, \;\; c > 0.$$

Removing the restriction on the constant $c > 0$ to be positive, we remove the absolute values in the above formula.
Thus, we obtain one-parameter family of explicit solutions:

$$y = cx - 1, \quad c \quad \text{is an arbitray constant.}$$

This is **an explicit solution of the given DE.**

Problem 1. Solve the DE

$$\frac{dy}{dx} = 3yx^2$$

by using the method of separation of variables.
Solution. The DE is in normal form. Next, we separate the variables x and y:

$$\frac{dy}{y} = 3x^2\,dx.$$

Integrating both sides we obtain:

$$\int \frac{dy}{y} = \int 3x^2\, dx$$

In order to solve the integral on the right-hand side, recall the **power rule**:

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c \text{ for } n \neq -1$$

In our case, $n = 2$, thus,

$$\begin{aligned} &\int 3x^2\, dx = 3\left(\frac{x^{2+1}}{3}\right) + c_2 \\ &\ln|y| + c_1 = x^3 + c_2 \\ &\ln|y| = x^3 + c. \end{aligned}$$

This is an implicit solution. In order to obtain an explicit solution we proceed as it follows:

$$e^{\ln|y|} = e^{x^3+c}$$

Since e^c is an arbitrary positive constant, we will denote it by $c > 0$. Also note that the e and the ln will cancel, the functions e° and $\ln(\circ)$ being a couple of direct and inverse functions, isolating y. Therefore,

$$|y| = e^c e^{x^3} = ce^{x^3}, \quad c > 0.$$

Dropping the condition $c > 0$ removes the absolute value in $|y|$ and yields **an explicit solution of the given DE :**

$$y = ce^{x^3}, \quad c \text{ is an arbitrary constant.}$$

Problem 2. Solve the DE

$$\sqrt[3]{x}\,dy - \sqrt{y}\,dx = 0$$

by the method of separation of the variables.

Note that the given DE is non-linear in y and non-linear in x, but it is separable. We separate the variables:

$$\sqrt[3]{x}\,dy = \sqrt{y}\,dx \quad \Rightarrow \frac{dy}{\sqrt{y}} = \frac{dx}{\sqrt[3]{x}} \quad \Rightarrow \quad \int y^{\frac{-1}{2}}\,dy = \int x^{\frac{-1}{3}}\,dx$$

Evaluating both integrals using the power rule yields:

$$\Rightarrow 2y^{\frac{1}{2}} = \frac{3}{2}x^{\frac{2}{3}} + c \quad \Rightarrow y^{\frac{1}{2}} = \frac{3}{4}x^{\frac{2}{3}}. + c$$

We obtain **an one-parameter family of explicit solutions:**

$$y = \left(\frac{3}{4}x^{\frac{2}{3}} + c\right)^2$$

Problem 3. Solve the DE

$$y' = e^{3x+2y}$$

by the method of separation of the variables.

Solution. The DE is given in a normal form and it is separable. First, we **replace the prime notation by the Leibnitz notation of a derivative**:

$$\frac{dy}{dx} = e^{3x+2y}.$$

The DE is separable because the right-hand side can be factored as

$$e^{3x+2y} = e^{3x}e^{2y}.$$

We separate the variables:

$$\frac{dy}{dx} = e^{3x}e^{2y} \quad \Rightarrow \frac{dy}{e^{2y}} = e^{3x}\,dx \quad \Rightarrow \int e^{-2y}\,dy = \int e^{3x}\,dx.$$

Useful integration formula:

$$\int e^{kx}\,dx = \frac{e^{kx}}{k} + c \text{ where } k \neq 0.$$

We integrate by using the above formula:

$$\frac{e^{-2y}}{-2} + c_1 = \frac{e^{3x}}{3} + c_2$$

Replacing $c_2 - c_1 = c$, where c is an arbitrary constant, we obtain

$$\frac{-e^{-2y}}{2} = \frac{e^{3x}}{3} + c \Rightarrow e^{-2y} = \frac{-2e^{3x}}{3} - 2c.$$

Note that $-2c$ is an arbitrary constant so, we replace the notation $-2c$ by c and therefore:

$$e^{-2y} = \frac{-2e^{3x}}{3} + c, \quad c \quad \text{an arbitrary constant.}$$

This is an implicit solution of the given DE. In order to obtain an explicit solution, we apply $\ln(\circ)$ to both sides in the above formula:

$$\begin{aligned}
&\ln(e^{-2y}) = \ln\left(c - \frac{2e^{3x}}{3}\right) \quad \Rightarrow -2y = \ln\left(c - \frac{2e^{3x}}{3}\right) \\
&\Rightarrow y = \frac{-1}{2}\ln\left(c - \frac{2e^{3x}}{3}\right) \quad \Rightarrow y = \ln\left(c - \frac{2e^{3x}}{3}\right)^{\frac{-1}{2}} \\
&y = \ln\frac{1}{\sqrt{c - \frac{2}{3}e^{3x}}}.
\end{aligned}$$

In view of this, explicit one-parameter family of solutions of the given DE is:

$$y = \ln\frac{1}{\sqrt{c - \frac{2}{3}e^{3x}}} \quad c \quad \text{constant.}$$

Problem 4. Solve the following differential equation:

$$yy' = \frac{y^2 - 1}{(x - 1)}.$$

Remark. General solution for a first-order DE means the set of all one-parameter family of solutions of the given first-order DE. Solve the given DE means find the general solution in an explicit form, if possible.

Solution of Problem 4. First, we represent the given DE in normal form (note that y is the dependent variable), replacing the prime notation by the Leibnitz notation of derivative:

$$\frac{dy}{dx} = \frac{y^2-1}{y(x-1)}.$$

The DE is separable because the right-hand side can be factored in a form:

$$\frac{y^2-1}{y(x-1)} = \left(\frac{y^2-1}{y}\right)\left(\frac{1}{x-1}\right).$$

Next, we separate the variables and integrate:

$$\frac{dy}{dx} = \frac{y^2-1}{y(x-1)} \quad \Rightarrow \frac{y}{y^2-1}\,dy = \frac{dx}{x-1}$$
$$\Rightarrow \int \frac{y}{y^2-1} dy = \int \frac{dx}{x-1}$$

In order to solve the integral on the right-hand side, recall that

$$\int \frac{1}{ax+b}\,dx = \frac{\ln|ax+b|}{a} + c \quad \text{where} \quad a \neq 0.$$

Therefore, integrating the right-hand side gives:

$$\ln|x-1| + c_2.$$

In order to solve the integral on the left-hand side, we integrate by using substitution. Make the substitution $v = y^2 - 1$. Then,

$$dv = 2ydy$$

$$\frac{dv}{2} = ydy$$

Hence,

$$\int \frac{y}{y^2-1}\,dy = \int \frac{dv}{2v} \quad \Rightarrow \frac{1}{2}\int \frac{dv}{v} = \frac{1}{2}\ln|v| + c_1 = \frac{\ln|y^2-1|}{2} + c_1.$$

Equating the left and right-hand sides together yields:

$$\ln|y^2-1| = 2\ln|x-1| + c \quad \Rightarrow e^{\ln|y^2-1|} = e^{2\ln|x-1|+c}$$
$$\Rightarrow e^{\ln|y^2-1|} = e^c e^{\ln(x-1)^2} \Rightarrow |y^2-1| = c(x-1)^2, \quad c > 0\,.$$

Removing the restriction $c > 0$ on the constant c will remove the absolute value in $|y^2 - 1|$ and from here:

$$y^2 - 1 = c(x-1)^2 \quad \Rightarrow \quad y^2 = 1 + c(x-1)^2,$$

where c is an arbitrary constant.
Implicit solution:

$$y^2 = 1 + c(x-1)^2, \quad c \quad \text{an arbitrary constant.}$$

Explicit solutions:

$$y_1 = \sqrt{1 + c(x-1)^2}, \quad y_2 = -\sqrt{1 + c(x-1)^2}, \quad c \text{ is an arbitrary constant.}$$

Problem 5. Find explicit solution of the following DE:

$$dx + e^{3x}dy = 0.$$

Remark. Equivalent representation of the given DE are

$$e^{3x}\frac{dy}{dx} + 1 = 0; \qquad \frac{dx}{dy} + e^{3x} = 0.$$

The DE is linear in y but non-linear in x.
Solution. Assuming y is the dependent variable, the normal form of the DE is

$$\frac{dy}{dx} = -e^{-3x}$$

and the separable differential form is:

$$dy = -e^{-3x}\,dx.$$

Integrating both sides

$$\int dy = \int -e^{-3x}dx \quad \Rightarrow \int dy = -\int e^{-3x}dx$$

we obtain the solution in an explicit form:

$$y = \frac{e^{-3x}}{3} + c, \quad c \quad \text{an arbitrary constant.}$$

Remark. The integration above is done by using the formula:

$$\int e^{kx}\,dx = \frac{e^{kx}}{k} + c \text{ where } k \neq 0.$$

Problem 6. Solve the given

$$\frac{dy}{dx} = \sin(5x) + \cos(3x).$$

Solution. The DE is in a normal form and obviously, it is separable. Then,

$$\int dy = \int [\sin(5x) + \cos(3x)]\ dx$$

Note that the following integration formulas hold:

$$\int \sin(kx)\,dx = -\frac{\cos(kx)}{k} + c$$

$$\int \cos(kx)\,dx = \frac{\sin(kx)}{k} + c,$$

where $k \neq 0$.
By using the above two integration formulas, we obtain **an explicit general solution**:

$$y = -\frac{\cos(5x)}{5} + \frac{\sin(3x)}{3} + c, \quad c \quad \text{an arbitrary constant.}$$

Problem 7. Solve the initial-value problem (IVP)

$$\sqrt{x}y' = (y^2 - 4)(x + 1), \quad y(0) = -3.$$

by using the method of separation of the variables.
Solution. **First, we find the general solution of the given DE by using separation of the variables.** The normal form of the DE by using Leibnitz notation of a derivative is:

$$\frac{dy}{dx} = \frac{(y^2 - 4)(x + 1)}{\sqrt{x}}.$$

The right-hand side is of the form $f(x)g(y)$ hence, the given DE is separable. We separate the variables:

$$\frac{(y^2 - 4)(x + 1)}{\sqrt{x}} = (y^2 - 4)\frac{x + 1}{\sqrt{x}}.$$

Separating the variables and integrating both sides gives:

$$\frac{dy}{dx} = (y^2 - 4)\frac{x + 1}{\sqrt{x}} \quad \Rightarrow \frac{dy}{y^2 - 4} = \frac{x + 1}{x^{\frac{1}{2}}}\,dx$$
$$\Rightarrow \int \frac{dy}{y^2 - 4} = \int \frac{x + 1}{x^{\frac{1}{2}}}\,dx\,.$$

We evaluate the integral on the left-hand side:

$$\int \frac{dy}{y^2 - 4} = \int \frac{dy}{(y - 2)(y + 2)}$$

by using the method of partial fractions (integration of rational functions):

$$\frac{1}{(y-2)(y+2)} = \frac{A}{y-2} + \frac{B}{y+2}$$
$$\Rightarrow \quad 1 = (y+2)A + (y-2)B.$$

With $y = 2$ we obtain $A = 1/4$ and when $y = -2$ we have $B = -1/4$. Therefore,

$$\frac{1}{(y-2)(y+2)} = \frac{1/4}{y-2} - \frac{1/4}{y+2}$$

Now evaluate $\int \frac{dy}{y^2-4} = \int \frac{dy}{(y-2)(y+2)}$ using the result of the partial fraction:

$$\int \frac{dy}{y^2-4} = \int \frac{dy}{(y-2)(y+2)} = \int \left(\frac{1/4}{y-2} - \frac{1/4}{y+2}\right) dy$$
$$\Rightarrow \frac{\ln|y-2|}{4} - \frac{\ln|y+2|}{4} + c_1 = \frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| + c_1$$

We evaluate the integral on the right-hand side:

$$\int \frac{x+1}{x^{\frac{1}{2}}}\, dx = \int \left(\frac{x}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}}}\right) dx = \int \left(x^{\frac{1}{2}} + x^{\frac{-1}{2}}\right) dx = \frac{2}{3}x^{\frac{2}{3}} + 2x^{\frac{1}{2}} + c_2$$

Equating the left and right-hand sides together:

$$\Rightarrow \frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| = \frac{2}{3}x^{\frac{2}{3}} + 2x^{\frac{1}{2}} + c \quad \Rightarrow \ln\left|\frac{y-2}{y+2}\right| = \frac{8}{3}x^{\frac{3}{2}} + 8x^{\frac{1}{2}} + c \quad \Rightarrow e^{\ln\left|\frac{y-2}{y+2}\right|} = ce^{\frac{8}{3}x^{\frac{3}{2}} + 8x^{\frac{1}{2}}}$$

Note that the notation e^c has been replaced by c, where $c > 0$ is an arbitrary positive constant. Then,

$$\Rightarrow \left|\frac{y-2}{y+2}\right| = ce^{\frac{8}{3}x^{\frac{3}{2}} + 8x^{\frac{1}{2}}}$$

and dropping the condition c to be positive we remove the absolute value on the left-hand side to obtain:

$$\frac{y-2}{y+2} = ce^{\frac{8}{3}x^{\frac{3}{2}} + 8x^{\frac{1}{2}}}$$
$$\Rightarrow y - 2 = (y+2)ce^{\frac{8}{3}x^{\frac{3}{2}} + 8x^{\frac{1}{2}}}$$
$$\Rightarrow y - 2 = yce^{\frac{8}{3}x^{\frac{3}{2}} + 8x^{\frac{1}{2}}} + 2ce^{\frac{8}{3}x^{\frac{3}{2}} + 8x^{\frac{1}{2}}}$$
$$\Rightarrow y(1 - ce^{\frac{8}{3}x^{\frac{3}{2}} + 8x^{\frac{1}{2}}}) = 2(1 + ce^{\frac{8}{3}x^{\frac{3}{2}} + 8x^{\frac{1}{2}}}).$$

Hence, the explicit solution of the given DE is:

$$y = 2\left(\frac{1+ce^{\frac{8}{3}x^{\frac{3}{2}}+8x^{\frac{1}{2}}}}{1-ce^{\frac{8}{3}x^{\frac{3}{2}}+8x^{\frac{1}{2}}}}\right)$$

where c is an arbitrary constant.
Second, we apply the given initial value condition. According to the given initial-value condition $y(0)=-3$, we are looking for a solution passing through the point $(0,-3)$. This means that for $x=0$ we have $y=-3$:

$$-3 = 2\frac{1+c}{1-c} \quad \Rightarrow \quad -3+3c = 2+2c \Rightarrow c = 5.$$

Hence, the unique solution of the given IVP is:

$$y = 2\left(\frac{1+5e^{\frac{8}{3}x^{\frac{3}{2}}+8x^{\frac{1}{2}}}}{1-5e^{\frac{8}{3}x^{\frac{3}{2}}+8x^{\frac{1}{2}}}}\right).$$

Losing of a Solution. Obviously, $y=-2$, $y=2$ are two constant solutions of the DE in the above problem. If $c=0$, we obtain $y=2$ but $y=-2$ cannot be obtained by using the above general solution. This effect is called: **Losing of a solution**. It can be explained by the fact that we divided by y^2-4, i.e., with the separable method of the solution process. **The solution $y=-2$ is called a singular solution.** However, we can represent the general solution in a different form:

$$y = 2\left(\frac{\frac{1}{c}+e^{\frac{8}{3}x^{\frac{3}{2}}+8x^{\frac{1}{2}}}}{\frac{1}{c}-e^{\frac{8}{3}x^{\frac{3}{2}}+8x^{\frac{1}{2}}}}\right)$$

Denoting the constant $\frac{1}{c}$ by c we obtain

$$y = 2\left(\frac{c+e^{\frac{8}{3}x^{\frac{3}{2}}+8x^{\frac{1}{2}}}}{c-e^{\frac{8}{3}x^{\frac{3}{2}}+8x^{\frac{1}{2}}}}\right).$$

Now, $y=-2$ is from the family of solutions, but $y=2$ is a singular solution.

Problem 8. Solve the DE

$$y\ln(x)x' = \left(\frac{y+1}{x}\right)^2.$$

Solution.

$$y\ln(x)\frac{dx}{dy} = \frac{(y+1)^2}{x^2} \quad \Rightarrow \int x^2\ln(x)\,dx = \int \frac{(y+1)^2}{y}\,dy.$$

Solving the integral on the right-hand side yields:

$$\int \frac{(y+1)^2}{y}\,dy = \int \frac{y^2+2y+1}{y}\,dy = \int (y+2+y^{-1})\,dy$$
$$= \frac{y^2}{2} + 2y + \ln|y| + c_1.$$

For the integral on the left-hand side, use the method of integration by parts:

$$\int x^2\ln(x)\,dx = \int \ln(x)d\frac{x^3}{3} = \frac{x^3\ln x}{3} - \int \frac{x^3}{3}d\ln x$$
$$\Rightarrow \frac{x^3\ln x}{3} - \frac{1}{3}\int x^3\frac{1}{x}\,dx$$
$$= \frac{x^3}{3}\ln(x) - \frac{1}{3}\int x^2\,dx$$
$$= \frac{x^3\ln(x)}{3} - \frac{1}{9}x^3 + c_2$$

Equating both sides gives an implicit solution (**a one-parameter family of implicit solutions**) of the given DE:

$$\Rightarrow \frac{y^2}{2} + 2y + \ln|y| = \frac{x^3\ln(x)}{3} - \frac{1}{9}x^3 + c$$

Problem 9. Solve the differential equation

$$e^x y\,dy + (e^{-y} + e^{-2x-y})\,dx = 0.$$

Solution. The DE is given in a differential form and it is separable. We separate the variables x and y:

$$e^x y dy = -e^{-y}(1+e^{-2x})dx$$
$$\Rightarrow ye^y dy = -e^{-x}(1+e^{-2x})dx$$
$$\Rightarrow \int ye^y dy = \int -e^{-x}(1+e^{-2x})dx.$$

Now, for the left-hand side, we integrate, using integration by parts:

$$\int ye^y dy = \int y de^y = ye^y - \int e^y \, dy = ye^y - e^y + c_1$$

To integrate the right-hand side, recall that

$$\int e^{kx} dx = \frac{e^{kx}}{k} + c \text{ when } k \neq 0.$$

Therefore,

$$\Rightarrow \int -e^{-x}(1 + e^{-2x}) \, dx = -\int (e^{-x} + e^{-3x}) \, dx = -\left(\frac{e^{-x}}{-1} + \frac{e^{-3x}}{-3}\right) + c_2$$
$$= e^{-x} + \frac{1}{3}e^{-3x} + c_2$$

Equating both sides:

$$ye^y - e^y + c_1 = e^{-x}\left(1 + \frac{1}{3}e^{-2x}\right) + c_2$$

Finally, we obtain the solution in an implicit form, i.e., we obtain an implicit one-parameter family of solutions:

$$(y-1)e^y = e^{-x}\left(1 + \frac{1}{3}e^{-2x}\right) + c.$$

Problem 10. Find **an implicit, and if possible an explicit solution** of the given first-order IVP (initial-value problem):

$$\frac{dx}{dt} = 3x^2 + 12, \qquad x\left(\frac{\pi}{8}\right) = 2.$$

Solution.

$$\frac{dx}{3(x^2+4)} = dt$$

Integrating both sides,

$$\int \frac{dx}{3(x^2+4)} = \int dt \quad \Rightarrow \int \frac{dx}{(x^2+4)} = 3\int dt$$

To solve the integral on the left-hand side, substitute $x = 2v$, then:

$$dx = 2dv$$

$$\int \frac{dx}{(x^2+4)} = \int \frac{2dv}{4v^2+4} = \int \frac{2dv}{4(v^2+1)} = \frac{1}{2}\int \frac{dv}{1+v^2}$$
$$= \frac{1}{2}\arctan(v) + c_1 = \frac{1}{2}\arctan(\frac{x}{2}) + c_1$$

Solving the integral on the right-hand side:

$$3\int dt = 3t + c_2$$

Equating both sides:

$$\frac{1}{2}\arctan\left(\frac{x}{2}\right) = 3t + c$$

In order to solve the IVP, we must first solve for c:

$$\arctan\left(\frac{x}{2}\right) = 6t + c$$

The IVP states that when $t = \frac{\pi}{8}, x = 2$.

$$\arctan(1) = \frac{3}{4}\pi + c$$
$$\frac{\pi}{4} = \frac{3\pi}{4} + c$$
$$c = \frac{-\pi}{2}$$

Hence, **an implicit solution to the given IVP** is:

$$\arctan\left(\frac{x}{2}\right) = 6t - \frac{\pi}{2}.$$

Remark. **Note that, in fact, the above relation is an explicit solution but in the variable t, in other words if we consider t as a dependent variable.** Applying $\tan(\circ)$ to both sides of the above expression in order to isolate x; this gives **an explicit solution to the given IVP in the variable x**:

$$\frac{x}{2} = \tan\left(6t - \frac{\pi}{2}\right) \quad \Rightarrow x = 2\tan\left(6t - \frac{\pi}{2}\right).$$

Problem 11. Solve the initial-value problem (IVP):

$$\sqrt{1-y^2}\,dx - \sqrt{1-x^2}\,dy = 0$$

$$y(0) = \frac{\sqrt{3}}{2}.$$

Solution.

$$\frac{dx}{\sqrt{1-x^2}} = \frac{dy}{\sqrt{1-y^2}} \quad \Rightarrow \int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dy}{\sqrt{1-y^2}} \quad \Rightarrow \arcsin(y) = \arcsin(x) + c$$

From the IVP,

$$\arcsin\left(\frac{\sqrt{3}}{2}\right) = \arcsin(0) + c \quad \Rightarrow \frac{\pi}{3} = 0 + c \quad \Rightarrow c = \frac{\pi}{3}.$$

Plugging c back into the general solution, we obtain:

$$\sin(\arcsin(y)) = \sin\left(\arcsin(x) + \frac{\pi}{3}\right) \quad \Rightarrow y = \sin\left(\arcsin(x) + \frac{\pi}{3}\right).$$

Applying the trigonometric identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ into the explicit solution gives the following equivalent form of the solution:

$$y = \sin(\arcsin(x))\cos\left(\frac{\pi}{3}\right) + \cos(\arcsin(x))\sin\left(\frac{\pi}{3}\right)$$

$$\Rightarrow y = \frac{x}{2} + \frac{\sqrt{3}}{2}\sqrt{1-\sin^2(\arcsin(x))}$$

$$\Rightarrow y = \frac{x}{2} + \frac{\sqrt{3}}{2}\sqrt{1-x^2}.$$

Remark. Recall several important trigonometric identities:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin^2\alpha + \cos^2\alpha = 1$$
$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$
$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2 - 1 = 1 - 2\sin^2\alpha$$

Problem 12. Solve the differential equation:

$$y' - 3y = 5y^2, \quad y = y(x).$$

Remark. In order to solve the given DE we apply separation of the variables.

Remark. Note that the DE is not only separable but Bernoulli's DE. This suggests another method of a solution (see Section 2.5) that we shall study later.

Remark. Note that the given DE is autonomous, i.e., the right-hand side $5y^2 + 3y$ depends only on the dependent variable y; not on x. The constant solutions $y = 0$ and $y = -\frac{3}{5}$ are attractors or repellers for all other non-constant solutions of the given DE when $x \to \infty$. This can be easily seen by computing the sign of the first derivative of a solution satisfying a given initial value condition say at $x = 0$.

Solution. The given DE is separable and autonomous. The constant solution $y = -\frac{3}{5}$ is an attractor to all other solutions, when $x \to \infty$. The explicit form the solution also conforms this fact.

$$\frac{dy}{dx} = 3y + 5y^2 \quad \Rightarrow \frac{dy}{3y + 5y^2} = dx.$$

We evaluate the indefinite integrals:

$$\Rightarrow \int \frac{dy}{3y + 5y^2} = \int dx.$$

Evaluating the right-hand side:

$$\int dx = x + c_2.$$

Evaluating the left-hand side:

$$\int \frac{dy}{3y + 5y^2} = \int \frac{dy}{y(3 + 5y)}.$$

In order to solve this integral, we use the method of partial fractions:

$$\frac{1}{y(3 + 5y)} = \frac{A}{y} + \frac{B}{3 + 5y}$$

$$1 = A(3 + 5y) + By$$

When $y = 0, A = \frac{1}{3}$.
When $y = \frac{-3}{5}, B = \frac{-5}{3}$.

$$\frac{1}{y(3+5y)} = \frac{1}{3y} - \frac{5}{3(3+5y)}$$

Therefore,

$$\int \frac{dy}{y(3+5y)} = \int \left(\frac{1}{3y} - \frac{5}{3(3+5y)} \right) dy$$
$$\Rightarrow \frac{1}{3} \int \frac{1}{y}\, dy - \frac{5}{3} \int \frac{1}{5y+3}\, dy$$

We solve this integral knowing that:

$$\int \frac{1}{ax+b}\, dx = \frac{1}{a} \ln|ax+b| + c \text{ for } a \neq 0.$$

Hence,

$$\Rightarrow \frac{1}{3} \int \frac{1}{y}\, dy - \frac{5}{3} \int \frac{1}{5y+3}\, dy \quad \Rightarrow \frac{1}{3} \ln|y| - \frac{5}{3}\left(\frac{1}{5}\right) \ln|5y+3| + c_1 \quad \Rightarrow \frac{1}{3} \ln\left|\frac{y}{5y+3}\right| + c_1$$

Equating the left and right-hand sides together, gives:

$$\frac{1}{3} \ln\left|\frac{y}{5y+3}\right| = \int dx = x + c \quad \Rightarrow \ln\left|\frac{y}{5y+3}\right| = 3x + c$$
$$\Rightarrow e^{\ln\left|\frac{y}{5y+3}\right|} = e^{3x+c} \quad \Rightarrow \left|\frac{y}{5y+3}\right| = e^c e^{3x}.$$

Denoting e^c by another constant $c > 0$ we obtain

$$\left|\frac{y}{5y+3}\right| = ce^{3x},\ c > 0 \text{ an arbitrary constant}$$

and without the restriction $c > 0$ the above formula takes the form

$$\frac{y}{5y+3} = ce^{3x}.$$

Solving with respect to y gives an explicit solution of the given DE:

$$y = 5ce^{3x}y + 3ce^{3x} \quad \Rightarrow y(1 - 5ce^{3x}) = 3ce^{3x} \quad \Rightarrow$$
$$y = \frac{3ce^{3x}}{1 - 5ce^{3x}}$$

Problem 13. Solve the DE, expressing your answer in explicit form:

$$\sqrt{y^2+1}\,dx = xy\,dy.$$

Solution. The given DE is separable.

$$\frac{y}{\sqrt{y^2+1}}\,dy = \frac{1}{x}\,dx \quad \Rightarrow \int \frac{y}{\sqrt{y^2+1}}\,dy = \int \frac{dx}{x}$$

Evaluating the integral on the right-hand side:

$$\int \frac{dx}{x} = \ln|x| + c_2$$

To evaluate the left-hand side, it is necessary to use the method of simple substitution, substituting $v = y^2 + 1$, then:

$$dv = 2y\,dy$$
$$\frac{dv}{2} = y\,dy$$

Therefore,

$$\int \frac{y}{\sqrt{y^2+1}}\,dy = \frac{1}{2}\int \frac{dv}{\sqrt{v}} = \frac{1}{2}\int v^{-\frac{1}{2}}\,dv$$
$$= v^{\frac{1}{2}} + c_1 = (y^2+1)^{\frac{1}{2}} + c_1 = \sqrt{y^2+1} + c_1.$$

Equating the left and right-hand sides:

$$\sqrt{y^2+1} + c = \ln|x| \quad \Rightarrow e^{\ln|x|} = e^{\sqrt{y^2+1}+c}$$
$$\Rightarrow |x| = e^c e^{\sqrt{y^2+1}},\ c > 0.$$

Therefore, the explicit general solution is

$$x = ce^{\sqrt{y^2+1}}$$

Note that the solution is written in the form $x = x(y)$. In other words, in the explicit form of the solution, x is the dependant variable and y is the independent variable! Also, you can verify that $x = ce^{\sqrt{y^2+1}}$ is the solution of the DE in order to see if the solution is correct.

Problem 14. Solve the given first-order IVP:

$$\sqrt{x^2+1}\,y' = xy^3, \quad y(0) = 2.$$

Solution. The given DE is separable. First, we solve the given DE:

$$\frac{dy}{y^3} = \frac{x}{\sqrt{x^2+1}}\,dx \quad \Rightarrow \int \frac{dy}{y^3} = \int \frac{x}{\sqrt{x^2+1}}\,dx$$

Solve the integral on the left-hand side using power rule:

$$\int \frac{dy}{y^3} = \int y^{-3}\,dy = \frac{-1}{2}y^{-2} + c_1$$

Solve the integral on the right-hand side by substituting $v = x^2 + 1$, then:

$$dv = 2x\,dx$$

$$\frac{dv}{2} = x\,dx$$

Therefore,

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int \frac{dv}{\sqrt{v}} = \frac{1}{2}\int v^{-\frac{1}{2}}\,dv$$
$$= v^{\frac{1}{2}} + c_2 = (x^2+1)^{\frac{1}{2}} + c_2 = \sqrt{x^2+1} + c_2.$$

Equating the right and left-hand sides gives:

$$\frac{-1}{2}y^{-2} + c_1 = \sqrt{x^2+1} + c_2 \quad \Rightarrow y^{-2} = -2\sqrt{x^2+1} + c.$$

Now we apply the given the given initial-value condition $y(0) = 2$ in order to determine the constant c:

$$2^{-2} = -2\sqrt{0^2+1} + c \quad \Rightarrow c = 2 + \frac{1}{4} = \frac{9}{4}$$

Plugging $c = 9/4$ back into the solution yields an implicit solution of the given IVP:

$$\frac{1}{y^2} = -2\sqrt{x^2+1} + \frac{9}{4}.$$

From the implicit solution we have two candidates for an explicit solution:

$$y_1(x) = \frac{1}{\sqrt{\frac{9}{4} - 2\sqrt{x^2+1}}}; \quad y_2(x) = -\frac{1}{\sqrt{\frac{9}{4} - 2\sqrt{x^2+1}}}.$$

Now, note that $y_2(x) = -\frac{1}{\sqrt{\frac{9}{4}-2\sqrt{x^2+1}}}$ does not satisfy the initial value condition $y(0) = 2$ hence, $y_2(x)$ is not a solution. In view of this

$$y_1(x) = \frac{1}{\sqrt{\frac{9}{4} - 2\sqrt{x^2+1}}}$$

is an explicit solution of the given IVP.

Problem 16. Solve the the given first-order IVP:

$$\cos(x)\,(e^{2y} - y)\frac{dy}{dx} = e^y \sin 2x, \quad y(0) = 0.$$

Solution. First we separate the variables.

$$\frac{e^{2y} - y}{e^y}\,dy = \frac{\sin(2x)}{\cos(2x)}\,dx$$

or which is equivalent to

$$\begin{aligned}(e^y - ye^{-y})\,dy &= 2\sin(x)\,dx \\ \int e^y\,dy - \int ye^{-y}\,dy &= 2\int \sin(x)\,dx.\end{aligned}$$

After the integration by parts, we obtain:

$$e^y + ye^{-y} + e^{-y} = -2\cos(x) + c$$

The initial condition $y = 0$ when $x = 0$ yields $c = 4$. Thus, **an implicit solution** of the guven IVP is

$$e^y + ye^{-y} + e^{-y} = 4 - 2\cos(x).$$

Problem 17. Solve the given IVP, expressing your answer in explicit form:

$$yx^2\frac{dy}{dx} = \sqrt[3]{y} - x^5\sqrt[3]{y}\ ,\ y(-1) = 1.$$

Solution.

$$\begin{aligned}&\Rightarrow yx^2\frac{dy}{dx} = \sqrt[3]{y}(1 - x^5) \\ &\Rightarrow yx^2\frac{dy}{dx} = y^{\frac{1}{3}}(1 - x^5) \\ &\Rightarrow y^{\frac{2}{3}}\,dy = \frac{1 - x^5}{x^2}\,dx \\ &\Rightarrow \int y^{\frac{2}{3}}\,dy = \int \frac{1 - x^5}{x^2}\,dx \\ &\Rightarrow \int y^{\frac{2}{3}}\,dy = \int (x^{-2} - x^3)\,dx \\ &\Rightarrow \frac{3}{5}y^{\frac{5}{3}} = -x^{-1} - \frac{x^4}{4} + c.\end{aligned}$$

Solving the IVP,

$$\Rightarrow \frac{3}{5}1^{\frac{5}{3}} = -(-1)^{-1} - \frac{(-1)^4}{4} + c$$
$$\frac{3}{5} = 1 - \frac{1}{4} + c \quad \Rightarrow c = \frac{3}{5} - \frac{3}{4} = \frac{-3}{20}.$$

Therefore,

$$\Rightarrow \frac{3}{5}y^{\frac{5}{3}} = \frac{-1}{x} - \frac{x^4}{4} - \frac{3}{20}$$
$$\Rightarrow y^{\frac{5}{3}} = -\frac{5}{3x} - \frac{5x^4}{12} - \frac{1}{4}$$
$$y = -\left(\frac{5}{3x} + \frac{5x^4}{12} + \frac{1}{4}\right)^{\frac{3}{5}}.$$

Problem 18. Solve the IVP, expressing your answer in explicit form:

$$\frac{dy}{dt} + y^2 = 1, \quad y(0) = 0.$$

Solution. **First, we find one-parameter family of solutions of the given DE:**

$$\frac{dy}{dt} = 1 - y^2 \quad \Rightarrow \frac{dy}{1-y^2} = dt \quad \Rightarrow \int \frac{dy}{1-y^2} = \int dt.$$

Solving the right-hand side:

$$\int dt = t + c_2$$

Solving the left-hand side using the method of partial fractions,

$$\begin{aligned} &\frac{1}{1-y^2} = \frac{1}{(1-y)(1+y)} = \frac{A}{1-y} + \frac{B}{1+y} \\ &\Rightarrow 1 = A(1+y) + B(1-y). \end{aligned}$$

When $y = -1$, $B = \frac{1}{2}$.
When $y = 1$, $A = \frac{1}{2}$.

$$\frac{1}{1-y^2} = \left(\frac{1}{2}\right)\frac{1}{1-y} + \left(\frac{1}{2}\right)\frac{1}{1+y}.$$

Therefore,

$$\begin{aligned} &\int \frac{dy}{1-y^2} = \int \left[\left(\frac{1}{2}\right)\frac{1}{1-y} + \left(\frac{1}{2}\right)\frac{1}{1+y}\right] dy \\ &\Rightarrow \frac{1}{2}\int \frac{1}{1-y}\,dy + \frac{1}{2}\int \frac{1}{1+y}\,dy \\ &\Rightarrow = -\frac{1}{2}\ln|1-y| + \frac{1}{2}\ln|1+y| + c_1 = \frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| + c_1. \end{aligned}$$

Equating the left and right-hand sides:

$$\begin{aligned} &\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| + c_1 = t + c_2 \quad \Rightarrow \ln\left|\frac{1+y}{1-y}\right| = 2t + c \\ &\Rightarrow e^{\ln\left|\frac{1+y}{1-y}\right|} = e^{2t+c} \quad \Rightarrow \left|\frac{1+y}{1-y}\right| = e^c e^{2t} \quad \Rightarrow \left|\frac{1+y}{1-y}\right| = ce^{2t}. \end{aligned}$$

Note that the notation e^c for the constant has been replaced by c, where $c > 0$ is an arbitrary positive constant. Because of this condition, we remove the absolute value and obtain the implicit solution:

$$\Rightarrow \frac{1+y}{1-y} = ce^{2t} \text{ , } c \text{ is an arbitrary constant.}$$

Second, we apply the given initial value condition to find a solution of the IVP: With $t = 0,\ y = 0)$ we obtain:

$$\frac{1+0}{1-0} = ce^{2\times 0} \quad \Rightarrow c = 1.$$

Therefore,

$$\Rightarrow \frac{1+y}{1-y} = e^{2t} \quad \Rightarrow 1 + y = (1-y)e^{2t} \quad \Rightarrow 1 + y = e^{2t} - e^{2t}y$$
$$\Rightarrow y(1 + e^{2t}) = e^{2t} - 1 \quad \Rightarrow y = \frac{e^{2t} - 1}{e^{2t} + 1}$$

is an explicit solution of the given IVP.

Problem 19. Solve explicitly the following IVP by using separation of variables.

$$(1 + x^4)dy + x(1 + 4y^2)dx = 0, \text{ y(1)=0 .}$$

Solution. **First, we find one-parameter family of solutions of the given DE:**

$$\frac{dy}{1+4y^2} = \frac{-xdx}{1+x^4} \quad \Rightarrow \int \frac{dy}{1+4y^2} = \int \frac{-xdx}{1+x^4}.$$

Evaluating the integral on the left-hand side:

$$\int \frac{dy}{1+4y^2}.$$

Substituting $v = 2y$ we obtain:

$$dv = 2\,dy$$
$$dy = \frac{1}{2}\,dv$$

Therefore,

$$\Rightarrow \frac{1}{2}\int \frac{dv}{1+v^2} = \frac{1}{2}\arctan(v) + c_1 \quad \Rightarrow \frac{1}{2}\arctan(2y) + c_1.$$

We evaluate the integral on the right-hand side:

$$\int \frac{-xdx}{1+x^4}$$

Substitute $u = x^2$, $du = 2x\,dx$ and from here

$$\frac{1}{2}\,du = x\,dx.$$

Hence,

$$\Rightarrow -\frac{1}{2}\int \frac{du}{1+u^2} \quad \Rightarrow -\frac{1}{2}\arctan(u) + c_2 = -\frac{1}{2}\arctan(x^2) + c_2.$$

Equating both sides gives:

$$\arctan(2y) + c_1 = -\frac{1}{2}\arctan(x^2) + c_2$$
$$\Rightarrow \tan(\arctan(2y)) = \tan(c - \frac{1}{2}\arctan(x^2))$$
$$\Rightarrow 2y = \tan(c - \frac{1}{2}\arctan(x^2))$$
$$\Rightarrow y = \frac{1}{2}\tan(c - \frac{1}{2}\arctan(x^2)).$$

Second, we apply the given initial value condition to find a solution of the IVP:

$$0 = \frac{1}{2}\tan\left(c - \frac{\pi}{8}\right) \quad \Rightarrow c = \frac{\pi}{8}$$

Thus, the explicit solution to the given IVP is:

$$y = \frac{1}{2}\tan\left(\frac{\pi}{8} - \frac{1}{2}\arctan(x^2)\right).$$

Problem 20. Solve the following equation by the separation of variables method. You may leave the solution in an implicit form.

$$y^2\,dx - 2y^2\,dy = dy$$

Solution. First, divide the entire expression by dy. This gives:

$$y^2\frac{dx}{dy} - 2y^2 = 1 \quad \Rightarrow y^2\frac{dx}{dy} = 1 + 2y^2.$$

Integrating both sides,

$$\begin{aligned}
&\Rightarrow \int dx = \int \left(\frac{1+2y^2}{y^2}\right)\,dy\\
&\Rightarrow x + c = \int \left(\frac{1}{y^2} + 2\right)\,dy\\
&\Rightarrow x + c = \int \left(y^{-2} + 2\right)\,dy\\
&\Rightarrow x + c = -y^{-1} + 2y\\
&\Rightarrow x + c = -\frac{1}{y} + 2y \Rightarrow x(y) = 2y - \frac{1}{y} + c.
\end{aligned}$$

Note that we got an explicit one-parameter family of solutions in $x = x(y)$.

Problem 21. Solve **the the first-order IVP**

$$\frac{dy}{dx} = e^{2x}e^{-y} + e^{3x}e^{-y}, \quad y(0) = 0,$$

expressing your answer in an explicit form.

Solution. First, we solve the given DE; or in other words, we find the general solution of the given DE, or in other words, we find the one-parameter family of solutions of the given DE. Factor out the e^{-y} to obtain:

$$\frac{dy}{dx} = e^{-y}(e^{2x} + e^{3x}).$$

Separating the variables and integrating both sides we obtain the solution in an explicit form:

$$\begin{aligned}
&\int e^y\,dy = \int (e^{2x} + e^{3x})\,dx \\
&\Rightarrow e^y = \frac{1}{2}e^{2x} + \frac{1}{3}e^{3x} + c \\
&\Rightarrow y = \ln\left(\frac{1}{2}e^{2x} + \frac{1}{3}e^{3x} + c\right).
\end{aligned}$$

In order to obtain the solution of the given first-order IVP we apply the given initial value condition:

$$e^0 = \frac{1}{2}e^0 + \frac{1}{3}e^0 + c \quad \Rightarrow c = 1 - \frac{5}{6} \quad \Rightarrow c = \frac{1}{6}.$$

Plugging c back into the general expression we obtain the solution of the IVP:

$$\begin{aligned}
&\Rightarrow e^y = \frac{1}{2}e^{2x} + \frac{1}{3}e^{3x} + \frac{1}{6} \\
&\Rightarrow y = \ln\left(\frac{1}{2}e^{2x} + \frac{1}{3}e^{3x} + \frac{1}{6}\right).
\end{aligned}$$

Remark. According to Poincare's Theorem (Theorem 1.2.1/ page 14/textbook (see also Theorem 1, page 19, the Teaching Material for WEEK 1 on Moodle)) because $f(x,y) = e^{-y}(e^{2x}+e^{3x})$ and $\partial f/\partial y = -e^{-y}(e^{2x}+e^{3x})$ are continuous in each rectangle containing the point $(0,0)$, the given IVP has a unique solution.

2.3. First Order Linear Differential Equations. Standard Form of a First Order Linear Differential Equation. Integrating Factor.

Consider a first-order linear DE written either by using Leibnitz notation or by using the prime notation of a derivative:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad \Leftrightarrow \quad a_1(x)y^{'} + a_0(x)y = g(x).$$

Assuming that $a_1(x) \neq 0$ and dividing both sides of the above DE by $a_1(x)$ we obtain the DE in **standard form** :

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)} \quad \Leftrightarrow \quad y^{'} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}.$$

If we denote for simplicity $p(x) = \frac{a_0(x)}{a_1(x)}$ and $f(x) = \frac{g(x)}{a_1(x)}$, then the **standard form** of the given DE is the following:

$$\frac{dy}{dx} + p(x)y = q(x) \quad \Leftrightarrow \quad y^{'} + p(x)y = q(x).$$

Representing the given DE in standard form is the first important step in **the method of integrating factor to solve first-order linear DEs** that is the basic topic of the present section. Here are some exercises on standard form.
Problem. Represent the DEs in **standard form**:

$$\textbf{(a)} \quad x^2y' + xy = 1.$$

Solution. Dividing both sides of the given DE by the coefficient of $y^{'}$ we obtain the DE in **Standard Form**:

$$x^2y' + xy = 1 \quad \Rightarrow y' + \frac{x}{x^2}y = \frac{1}{x^2} \quad \Rightarrow y' + \frac{1}{x}y = \frac{1}{x^2}.$$

$$\textbf{(b)} \quad (x+2)^2\frac{dy}{dx} = 5 - 8y - 4xy.$$

Solution. First, taking into account that y is the dependent variable we represent the given first-order linear in y DE in the form

$$a_1(x)y^{'} + a_0(x)y = g(x).$$

We have

$$(x+2)^2\frac{dy}{dx} = 5 - 8y - 4xy \quad \Rightarrow (x+2)^2\frac{dy}{dx} = 5 - (8+4x)y$$
$$\Rightarrow (x+2)^2\frac{dy}{dx} + (8+4x)y = 5.$$

Next, dividing both sides by the coefficient $a_1(x) = (x+2)^2$ of y' we obtain the standard form of the given DE:

$$\frac{dy}{dx} + \frac{8+4x}{(x+2)^2}y = \frac{5}{(x+2)^2}.$$

The standard form can be further simplified observing that $8+4x$ can be factored to $4(x+2)$:

$$\frac{dy}{dx} + \frac{4}{x+2}y = \frac{5}{(x+2)^2}.$$

(c) $x\dfrac{dy}{dx} - y = x^2 \sin x.$

Solution.

$$x\frac{dy}{dx} - y = x^2 \sin x \quad \Rightarrow \frac{dy}{dx} - \frac{1}{x}y = x \sin x$$

(d) $\cos^2 x \sin x \dfrac{dy}{dx} + (\cos^3 x)y = 1.$

Solution.

$$\cos^2 x \sin x\frac{dy}{dx} + (\cos^3 x)y = 1 \quad \Rightarrow \frac{dy}{dx} + \frac{\cos^3 x}{\cos^2 x \sin x}y = \frac{1}{\cos^2 x \sin x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{\cos x}{\sin x}y = \frac{1}{\cos^2 x \sin x}$$

(e) $y\,dx = (ye^y - 2x)\,dy$

Solution. Assuming y dependent variable, the DE takes the form

$$(ye^y - 2x)\frac{dy}{dx} = y \quad \Leftrightarrow \quad (ye^y - 2x)\,y' = y$$

and in view of this we conclude that it is non-linear in y. Hence, **we can not talk about standard form in y because the DE is non-linear in y.**

However, assuming that x is the dependent variable the DE is linear in x and we can represent the DE in standard form:

$$\frac{dx}{dy} + \frac{2}{y}x = e^y \quad \Leftrightarrow \quad x^{'} + \frac{2}{y}x = e^y.$$

Remark. **Homogeneous first-order linear DE** is a DE with a right-hand side 0:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \quad \Rightarrow \quad \frac{dy}{dx} + p(x)y = 0 \quad \textbf{(standard form)}.$$

Note that **each homogeneous first-order DE is separable**:

$$\frac{dy}{dx} + p(x)y = 0 \quad \Leftrightarrow \quad \frac{dy}{y} = -p(x)dx$$

and can be solved by using the method of separating the variables, also.
Non-homogeneous first-order linear DE is a DE with non-zero right-hand side:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),\; g(x) \neq 0 \;\Rightarrow\; \frac{dy}{dx} + p(x)y = f(x) \quad \textbf{(standard form)}.$$

In general, a non-homogeneous first-order linear DE is not separable. For example, the first-order linear DE:

$$\frac{dy}{dx} - y = x \quad \Leftrightarrow \quad \frac{dy}{dx} = y + x$$

is not separable because $y + x$ can not be represented in *a product form* $r(x)s(y)$, where $r(x)$ is a function only in x and $s(y)$ is a function only in y.
Suppose that $x + y = r(x)s(y)$. Substituting $y = 0$ we obtain $x = r(x)s(0)$ hence, $r(x) = ax$, for some number a. Analogously, $s(y) = by$ for some number b and we should have $x + y = (ab)xy$ that is not possible.

Method of Integrating Factor
to solve First-Order Linear DEs

We solve **first-order linear DE** by using **the method of integrating factor**. Note that by using **the method of integrating factor** we find **the general solution** of a given first-order linear DE, in other words, **the set of all solutions** in **explicit form**. The **general solution** of a **first-order linear DE** is **one-parameter family of functions** in other words **a set of functions depending on one constant (one parameter)**.
Here are **the basic steps of the method of integrating factor:**

1. Represent the given DE in **standard form**:

$$y' + p(x)y = f(x).$$

2. Evaluate the **integrating factor**:

$$e^{\int p(x)\,dx}.$$

3. Multiply both sides of the standard form $y' + p(x)y = f(x)$ by the integrating factor $e^{\int p(x)\,dx}$. This gives the DE:

$$y'e^{\int p(x)\,dx} + p(x)e^{\int p(x)\,dx}y = e^{\int p(x)\,dx}f(x).$$

   Now, the left-hand side of the above DE:

$$y'e^{\int p(x)\,dx} + p(x)e^{\int p(x)\,dx}y$$

   is **a complete derivative** of the function $ye^{\int p(x)\,dx}$. In other words, by using the product rule for differentiation we obtain:

$$\frac{d}{dx}\left[ye^{\int p(x)\,dx}\right] = y'e^{\int p(x)\,dx} + p(x)e^{\int p(x)\,dx}y.$$

4. Hence,

$$[ye^{\int p(x)\,dx}]' = e^{\int p(x)\,dx}f(x).$$

5. Integrating both sides of the above equation yields:

$$\int [ye^{\int p(x)\,dx}]'dx = \int e^{\int p(x)\,dx}f(x)\,dx$$

$$ye^{\int p(x)\,dx} = \int e^{\int p(x)\,dx}f(x)\,dx + c.$$

6. Isolating y gives **the general solution** of the given DE that is **a one-parameter family of solutions**:

$$y = e^{-p(x)\,dx}\int e^{\int p(x)\,dx}f(x)\,dx + ce^{-\int p(x)\,dx}.$$

   Note that the method of integrating factor gives always explicit solutions.

Transient Term and Steady State Term of a Solution of a DE. Transient term (transient part) $y_t(x)$ of a solution of a given DE is a term such that $\lim_{x\to\infty} y_t(x) = 0$, in other words a term from the solution that disappears for a big time x. A term that is not transient is a steady state term of the solution.

Example. Find the general solution of the DE

$$x\frac{dy}{dx} + 4y = x^3 - x.$$

Determine wether there is any transient term in the solution.
Remark. The given first-order DE is linear in y hence, we can apply the method of integrating factor to solve it.
Solution. Step 1. First, transform the equation into standard form:

$$\frac{dy}{dx} + \frac{4}{x}y = x^2 - 1.$$

Step 2. Determine $p(x) = \frac{4}{x}$. Then, compute the integrating factor:

$$e^{p(x)\,dx} = e^{\int \frac{4}{x}\,dx} = e^{4\ln|x|} = e^{\ln x^4} = x^4.$$

Step 3. Multiply the standard form $\frac{dy}{dx} + \frac{4}{x}y = x^2 - 1$ by the integrating factor to obtain:

$$x^4y' + 4x^3y = x^6 - x^4.$$

Step 4. From the product rule, the left-hand side of the above DE is a complete derivative:

$$(x^4y)' = x^6 - x^4.$$

Step 5. We integrate both sides of the DE in Step 4:

$$\int (x^4y)'\,dx = \int (x^6 - x^4)\,dx \quad \Rightarrow x^4y = \frac{x^7}{7} - \frac{x^5}{5} + c.$$

Step 6. Solving the above equation for y, i.e., isolating y gives **the general solution (a one-parameter family of solutions) of the DE in explicit form**:

$$\Rightarrow y = \frac{x^3}{7} - \frac{x}{5} + \frac{c}{x^4}.$$

Obviously, $y_t(x) = c/x^4$ is the transient term of the solution. The steady-state term of the solution is $y_s(x) = x^3/7 - x/5$

Remark. Note that **the method of integrating factor gives the solution in explicit form**. Also, you can always **check if the solution is correct**.

$$y' = \frac{3}{7}x^2 - \frac{1}{5} - \frac{4c}{x^5}$$
$$xy' + 4y = x(\frac{3}{7}x^2 - \frac{1}{5} - \frac{4c}{x^5}) + 4(\frac{x^3}{7} - \frac{x}{5} + \frac{c}{x^4})$$
$$= \frac{3}{7}x^3 - \frac{1}{5}x - \frac{4c}{x^4} + \frac{4}{7}x^3 - \frac{4}{5}x + \frac{4c}{x^4} = x^3 - x$$

Therefore, we have verified that the solution is correct.

Problem 1. Find the general solution of the DE

$$(1 + x)y' - xy = 1.$$

Determine the transient term and the steady-state term in the solution.

Solution. First, standard form,

$$y' - \frac{x}{1+x}y = \frac{1}{1+x}$$

We determine:

$$p(x) = -\frac{x}{1+x}.$$

We compute the integrating factor:

$$e^{\int -\frac{x}{1+x}\,dx} = e^{\int -\frac{(1+x)-1}{1+x}\,dx} = e^{\int (-1+\frac{1}{1+x})\,dx}$$
$$= e^{-x+\ln(1+x)} = e^{-x+\ln(1+x)} = e^{-x}e^{\ln(1+x)} = (1+x)e^{-x}.$$

We multiply the standard form by the integrating factor $(1 + x)e^{-x}$:

$$(1 + x)e^{-x}y' - xe^{-x}y = e^{-x}.$$

By using the product rule we obtain:

$$[(1 + x)e^{-x}y]' = e^{-x}.$$

Integrating both sides gives:

$$\int [(1+x)e^{-x}y]'\,dx = \int e^{-x}\,dx.$$

$$(1+x)e^{-x}y = -e^{-x} + c$$

The general solution (a one-parameter family of solutions) is:

$$y = -\frac{1}{1+x} + \frac{ce^{x}}{1+x}.$$

Obviously, the transient term of the solution is is $y_t(x) = 1/(1+x)$ and the steady-state term is $ce^{x}/(1+x),\ c \neq 0$.

Problem 2. Find the general solution of the DE

$$(x+1)y' + (2x+3)y = xe^{-2x}.$$

Solution.
In standard form,

$$y' + \frac{2x+3}{x+1}y = \frac{x}{x+1}e^{-2x}.$$

Integrating factor:

$$e^{\int \frac{2x+3}{x+1}\,dx} = e^{\int (2+\frac{1}{x+1})\,dx} = e^{2x}e^{\ln(x+1)} = (x+1)e^{2x}.$$

Multiplying the equation that is in standard form by the integrating factor, we obtain:

$$\begin{aligned}
&(x+1)e^{2x}y' + (2x+3)e^{2x}y = x \\
&\Rightarrow \int [(x+1)e^{2x}y]'\,dx = \int x\,dx \\
&\Rightarrow (x+1)e^{2x}y = \frac{x^2}{2} + c \\
&\Rightarrow y = \frac{x^2e^{-2x}}{2(x+1)} + \frac{ce^{-2x}}{x+1}.
\end{aligned}$$

Problem 3. Find the general solution of the DE

$$\frac{dQ}{dv} + vQ = Q + v - 1.$$

Solution.
In standard form,

$$Q' + (v-1)Q = v - 1.$$

Integrating factor:

$$e^{(v-1)\,dv} = e^{\frac{v^2}{2}-v}.$$

Multiplying the equation that is in standard form by the integrating factor, we obtain:

$$\begin{aligned}
&e^{\frac{v^2}{2}-v}Q' + (v-1)e^{\frac{v^2}{2}-v}Q = e^{\frac{v^2}{2}-v}(v-1) \\
&\Rightarrow \int \left[e^{\frac{v^2}{2}-v}Q\right]' dv = \int (v-1)e^{\frac{v^2}{2}-v\,dv} \\
&\Rightarrow e^{\frac{v^2}{2}-v}Q = \int (v-1)e^{\frac{v^2}{2}-v\,dv}
\end{aligned}$$

We substitute $u = \frac{v^2}{2} - v$ in order to evaluate the integral. Then,

$$\begin{aligned}
&u = \frac{v^2}{2} - v \quad \Rightarrow \quad du = (v-1)dv \\
&\int (v-1)e^{\frac{v^2}{2}-v}\,dv = \int e^u\,du = e^u + c = e^{\frac{v^2}{2}-v} + c \\
&\Rightarrow e^{\frac{v^2}{2}-v}Q = e^{\frac{v^2}{2}-v} + c.
\end{aligned}$$

The explicit general solution (a one-parameter family of solutions) is:

$$Q = 1 + ce^{-\frac{v^2}{2}-v}.$$

Verify that $Q(v)$ is the solution of the given DE.
Problem 4. Find the general solution of the DE

$$(x^2-1)\frac{dy}{dx} + 2y = (x+1)^2.$$

Solution. In standard form,

$$y' + \frac{2}{x^2-1}y = \frac{x+1}{x-1}.$$

Determining the integrating factor:

$$e^{\int \frac{2}{x^2-1}\,dx}.$$

In order to solve $\int \frac{2}{x^2-1}\,dx$, note that x^2-1 can be factored to $(x-1)(x+1)$. We solve this integral by using partial fractions:

$$\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$
$$\Rightarrow 2 = A(x+1) + B(x-1).$$

When $x = 1, A = 1$.
When $x = -1, B = -1$.
Then,

$$\int \frac{2}{x^2-1}\,dx = \int \left(\frac{1}{x-1} - \frac{1}{x+1}\right)\,dx$$
$$\Rightarrow \ln|x-1| - \ln|x+1| = \ln\left|\frac{x-1}{x+1}\right|.$$

Therefore, after evaluating the integral, the integrating factor is:

$$e^{\int \frac{2}{x^2-1}\,dx} = e^{\ln|x-1| - \ln|x+1|} = \frac{x-1}{x+1},$$

assuming that $\frac{x-1}{x+1} > 0$.
Multiplying the DE in standard form by the integrating factor yields

$$\frac{x-1}{x+1}y' + \frac{2}{(x+1)^2}y = 1$$
$$\Rightarrow \left[\frac{x-1}{x+1}y\right]' = 1$$

Integrating both sides of this expression gives:

$$\int \left[\frac{x-1}{x+1}y\right]'\,dx = \int 1\,dx$$
$$\Rightarrow \frac{x-1}{x+1}y = x + c$$
$$\Rightarrow y = \frac{x(x+1)}{x-1} + \frac{c(x+1)}{x-1}.$$

Problem 5. Find the general solution of the DE

$$\cos^2(x)\sin(x)\frac{dy}{dx} + \cos^3(x)y = 1\,.$$

Solution. In standard form,

$$\frac{dy}{dx} + \frac{\cos x}{\sin x}y = \frac{1}{\cos^2(x)\sin(x)}.$$

Integrating factor:

$$e^{\int \frac{\cos x}{\sin x}\,dx} = e^{\int \cot x}y = e^{\ln \sin x} = \sin x.$$

Multiplying the DE in standard form by the integrating factor,

$$\sin(x)y' + \cos(x)y = \frac{1}{\cos^2(x)}.$$

Integrating both sides,

$$\int [\sin(x)y]'\,dx = \int \frac{1}{\cos^2(x)}\,dx$$
$$\sin(x)y = \tan(x) + c.$$

The general solution is:

$$y = \frac{1}{\cos x} + \frac{c}{\sin x}\,.$$

Verify the solution.

Problem 6. Solve the Initial Value Problem:

$$y\frac{dx}{dy} - x = 2y^2, \quad y(1) = 5\,.$$

Solution. Note that the DE is linear in x and non-linear in y. Therefore, in order to apply the method of integrating factor, we assume x dependent variable, i.e., $x = x(y)$.

In standard form,

$$x^{'} - \frac{1}{y}x = 2y.$$

Integrating factor:

$$e^{\int -\frac{1}{y}\,dy} = e^{-\ln y} = e^{\ln \frac{1}{y}} = \frac{1}{y}.$$

Then,

$$\frac{1}{y}x' - \frac{1}{y^2}x = 2$$
$$[\frac{1}{y}x]' = 2$$
$$\Rightarrow \int \left[\frac{1}{y}x\right]' dy = \int 2\,dy$$
$$\frac{1}{y}x = 2y + c$$

The general solution is:

$$x = 2y^2 + cy.$$

We determine c by using the given initial-value condition $y(1) = 5$, i.e., $x = 1, y = 5$:

$$1 = 50 + 5c \quad \Rightarrow c = \frac{-49}{5}$$

The solution to the IVP is:

$$x = 2y^2 - \frac{49}{5}y.$$

The solution of the given IVP is unique which makes sense according to Theorem 1.2.1/ page 14/textbook (see also Theorem 1, page 19, the Teaching Material for WEEK 1 on Moodle).

Problem 7. Find the general solution of the DE

$$xy' + (1 + x)y = e^{-x}\sin(2x)\,.$$

Solution. In standard form,

$$y' + \left(1 + \frac{1}{x}\right)y = \frac{e^{-x}}{x}\sin(2x).$$

Integrating factor:

$$e^{\left(1+\frac{1}{x}\right)dx} = xe^x.$$

Then,

$$\begin{aligned}
&xe^xy' + (x+1)e^xy = \sin(2x) \\
&\Rightarrow \int [xe^xy]'\,dx = \int \sin(2x)\,dx \\
&\Rightarrow xe^xy = \frac{-\cos(2x)}{2} + c \\
&\Rightarrow y = -\frac{e^{-x}\cos(2x)}{2x} + \frac{ce^{-x}}{x}.
\end{aligned}$$

This is the general solution of the given DE.

Problem 8. Find the general solution of

$$y\,dx = (ye^y - 2x)\,dy\,.$$

Solution. The DE is linear in x and non-linear in y. Hence, we apply the integrating factor method for $x = x(y)$, assuming x dependent variable. In standard form,

$$x' + \frac{2}{y}x = e^y$$

Integrating factor:

$$e^{\int \frac{2}{y}\,dy} = e^{2\ln y} = e^{\ln y^2} = y^2\,.$$

Then,

$$\begin{aligned}
&y^2x' + 2yx = y^2e^y \\
&\Rightarrow [y^2x]' = y^2e^y \\
&\Rightarrow \int [y^2x]'\,dy = \int y^2e^y\,dy \\
&\Rightarrow y^2x = \int y^2e^y\,dy.
\end{aligned}$$

To evaluate the integral on the right-hand side, use integration by parts:

$$\begin{aligned}
&\int y^2e^y\,dy = \int y^2de^y = y^2e^y - \int e^y\,dy^2 \\
&\Rightarrow y^2e^y - 2\int e^yy\,dy = y^2e^y - 2\int yde^y \\
&\Rightarrow y^2e^y - 2ye^y + 2\int e^y\,dy = y^2e^y - 2ye^y + 2e^y + c \\
&\Rightarrow (y^2 - 2y + 2)e^y + c.
\end{aligned}$$

Therefore,

$$\begin{aligned} y^2 x &= (y^2 - 2y + 2)e^y + c \\ \Rightarrow x &= \left(1 - \frac{2}{y} + \frac{2}{y^2}\right) e^y + \frac{c}{y^2} \end{aligned}$$

Verify the solution.
Problem 9. Solve the DE

$$\frac{dA}{dt} + \frac{2}{50+t} A = 3\,.$$

This DE is obtained from a Mathematical Model of Mixing of Solutions.

Solution. **Note that the given DE is linear in A but it is not separable.** The DE is already given in standard form and it is linear in $A = A(t)$. We solve it by using the method of integrating factor for linear first-order DE:

$$\begin{aligned} &p(t) = \frac{2}{50+t} \\ &\Rightarrow e^{\int \frac{2}{50+t}\,dt} = e^{2\ln(50+t)} = e^{\ln(50+t)^2} = (50+t)^2. \end{aligned}$$

Then,

$$\begin{aligned} &(50+t)^2 A' + 2(50+t)A = 3(50+t)^2 \\ &\Rightarrow \int [(50+t)^2 A]'\,dt = \int 3(50+t)^2\,dt. \end{aligned}$$

Solving the integral on the left gives:

$$\int [(50+t)^2 A]'\,dt = (50+t)^2 A\,.$$

Solving the integral on the right requires a simple substitution:
Substitute $v = 50 + t$. Then, $dv = dt$:

$$\int 3(50+t)^2\,dt = 3\int v^2\,dv = \frac{3v^3}{3} + c = (50+t)^3 + c.$$

Equating the left and right-hand sides yields:

$$\Rightarrow (50+t)^2 A = (50+t)^3 + c$$
$$\Rightarrow A = 50 + t + \frac{c}{(50+t)^2}.$$

Question. Determine **the transient term** and **the steady-state term** of the above solution.

Problem 10. Solve the first-order DE

$$xy' + (1+x)y = e^{-x}\sin(2x).$$

Solution. In standard form,

$$y' + \left(1 + \frac{1}{x}\right) y = \frac{e^{-x}}{x}\sin(2x).$$

Integrating factor:

$$e^{\left(1+\frac{1}{x}\right)dx} = e^{x+\ln x} = e^x e^{\ln x} = xe^x.$$

Then,

$$xe^x y' + e^x(x+1)y = \sin(2x)$$
$$\Rightarrow [xe^x y]' = \sin(2x)$$
$$\Rightarrow \int [xe^x y]'\, dx = \int \sin(2x)\, dx$$
$$\Rightarrow xe^x y = -\frac{\cos(2x)}{2} + c$$
$$\Rightarrow y = -\frac{e^{-x}\cos(2x)}{2x} + \frac{ce^{-x}}{x}.$$

Problem 11. Find the general solution of the DE

$$\frac{dy}{dx} - 2y = 2e^{2x}\cos x.$$

Solution. The DE is given in standard form. Integrating factor: $e^{\int -2\, dx} = e^{-2x}$. Then,

$$e^{-2x}y' - 2e^{-2x}y = 2\cos x$$

$$
\begin{aligned}
&\Rightarrow [e^{-2x}y]' = 2\cos x \\
&\Rightarrow \int [e^{-2x}y]'\,dx = \int 2\cos x\,dx \\
&\Rightarrow e^{-2x}y = 2\sin x + c \\
&\Rightarrow y = 2e^{2x}\sin x + ce^{2x}\,.
\end{aligned}
$$

Verify the solution.

Problem 12. Solve the DE by using the method of integrating factor:

$$\frac{dy}{dx} - \frac{y}{x} = 1.$$

Solution. The equation is in standard form.
Integrating factor:

$$e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}.$$

Then,

$$
\begin{aligned}
&\frac{1}{x}y' - \frac{1}{x^2}y = \frac{1}{x}. \\
&\Rightarrow \left[\frac{1}{x}y\right]' = \frac{1}{x}. \\
&\Rightarrow \int \left[\frac{1}{x}y\right]' dx = \int \frac{1}{x}\,dx \\
&\Rightarrow \frac{1}{x}y = \ln x + c \\
&\Rightarrow y = x\ln x + c\,x\,.
\end{aligned}
$$

Verify the solution.

Problem 13. Solve the DE:

$$\frac{dy}{dx} - 2y = e^x.$$

Solution. The given equation is linear in y and it is in standard form.
Integrating factor: $e^{\int -2\,dx} = e^{-2x}$.
Then,

$$
\begin{aligned}
&[e^{-2x}y]' = e^{-x} \\
&\Rightarrow \int [e^{-2x}y]'\,dx = \int e^{-x}\,dx \\
&\Rightarrow e^{-2x}y = -e^{-x} + c \\
&\Rightarrow y = -e^x + ce^{2x}\,.
\end{aligned}
$$

Verify the solution.
Problem 14. Solve the IVP:

$$xy' + y = \cos x,\ y(\pi) = 1$$

Solution. The DE is linear in y. In standard form:

$$y' + \frac{1}{x}y = \frac{\cos x}{x}$$

Integrating factor:

$$e^{\int \frac{1}{x}\,dx} = e^{\ln|x|} = x\,.$$

Then,

$$\begin{aligned} & xy' + y = \cos x \\ \Rightarrow \quad & \int [xy]'\,dy = \int \cos x\,dx \\ \Rightarrow \quad & xy = \sin(x) + c\,. \end{aligned}$$

Finding c:

$$(\pi)(1) = \sin(\pi) + c \quad \Rightarrow c = \pi\,.$$

Hence,

$$\begin{aligned} & xy = \sin(x) + \pi \\ & \Rightarrow y = \frac{\sin(x)}{x} + \frac{\pi}{x} = \frac{\sin(x)+\pi}{x}\,. \end{aligned}$$

Problem 15. Solve the DE:

$$(x+1)y' + xy = (x+1)^3.$$

Solution. The DE is linear in y. In standard form,

$$y' + \frac{x}{x+1}y = (x+1)^2\,.$$

To find the integrating factor, we must evaluate $e^{\int \frac{x}{x+1}\,dx}$. We first break down the fraction by performing long division. We then evaluate the integral:

$$e^{\int \frac{x}{x+1}\,dx} = e^{\left(-\frac{1}{x+1}+1\right)\,dx} = e^{-\ln|x+1|+x} = \frac{e^x}{x+1}\,.$$

Remark. Note that long division can only be used when the highest power in the numerator is greater than or equal to the highest power in the denominator. In this case,the highest power for both the numerator and the denominator is one, making long division possible. If this was not the case, try either the method of partial fractions or completing the square.
Then, we multiply the DE in standard form by the integrating factor and we integrate:

$$\begin{aligned}
&\frac{e^x}{x+1}y' + \frac{xe^x}{(x+1)^2}y = e^x(x+1)\\
&\Rightarrow \int \left[\frac{e^x y}{x+1}\right]' dy = \int e^x(x+1)\,dx\\
&\Rightarrow \frac{e^x y}{x+1} = \int (xe^x + e^x)\,dx\\
&\Rightarrow \frac{e^x y}{x+1} = \int xe^x\,dx + \int e^x\,dx\,.
\end{aligned}$$

The $\int xe^x\,dx$ can be solved using integration by parts. Thus,

$$\begin{aligned}
&\frac{e^x y}{x+1} = xe^x - e^x + e^x + c\\
&\Rightarrow \frac{e^x y}{x+1} = xe^x + c\\
&\Rightarrow y = \frac{xe^x(x+1)}{e^x} + \frac{c(x+1)}{e^x}\\
&\Rightarrow y = x(x+1) + ce^{-x}(x+1).
\end{aligned}$$

In conclusion:
- **By the method of separation of the variables, we can solve first-order, linear or non-linear DEs. However, the DEs must be separable.**
- **By the method of integrating factor, we solve first-order, linear DEs. Many linear first-order DE are not separable so, the method described in this section has its own value.**

Problem 16. Let us go back and solve a non-linear, separable, first-order DE:

$$(\sqrt{x} + x)\,dy - (\sqrt{y} + y)\,dx = 0.$$

Solution. Note that the DE is non-linear and thus, the integrating factor method is not applicable! Therefore, its is separable and we separate the variables in order to solve it:

$$\int \frac{dy}{\sqrt{y} + y} = \int \frac{dy}{\sqrt{x} + x}.$$

To solve the integral on the left-hand side, We substitute $y = v^2$ in order to solve the integral on the left-hand side:

$$y = v^2 \quad \Rightarrow \quad dy = 2v\,dv, \quad \sqrt{y} = v.$$

Therefore,

$$\begin{aligned}
&\int \frac{dy}{\sqrt{y}+y} = \int \frac{2v\,dv}{v+v^2} = 2\int \frac{dv}{1+v} \\
&\Rightarrow 2\ln(1+v) = 2\ln(1+\sqrt{y}) + c_1.
\end{aligned}$$

Using the same approach, we solve the integral on the right-hand side:

$$\int \frac{dx}{\sqrt{x}+x} = 2\ln(1+\sqrt{x}) + c_2.$$

Equating both sides gives:

$$\begin{aligned}
&2\ln(1+\sqrt{y}) = 2\ln(1+\sqrt{x}) + c \\
&\Rightarrow \ln(1+\sqrt{y}) = \ln(1+\sqrt{x}) + c \\
&\Rightarrow e^{\ln(1+\sqrt{y})} = e^{\ln(1+\sqrt{x})+c} \\
&\Rightarrow e^{\ln(1+\sqrt{y})} = e^c e^{\ln(1+\sqrt{x})} \\
&\Rightarrow 1+\sqrt{y} = c(1+\sqrt{x}) \Rightarrow \sqrt{y} = c(1+\sqrt{x}) - 1 \\
&\Rightarrow y = \left[c(1+\sqrt{x}) - 1\right]^2.
\end{aligned}$$