Applied Ordinary Differential Equations, ENGR-213, Section XX, Fall 2015

1.1. Definitions and Terminology

The derivative $\frac{dy}{dx}$ of a function $y = f(x)$ is another function $f^{'}(x)$ found by using appropriate rules for differentiation. For example, the function $y = e^{x^3}$ is differentiable on $(-\infty, \infty)$ and its derivative is

$$\frac{dy}{dx} = 3x^2 e^{x^3}.$$

If we replace e^{x^3} by y on the right-hand side of the above equation, we obtain another equation including the function y and its derivative

$$\frac{dy}{dx} = 3x^2 y$$

or equivalently by using the prime notation of derivative

$$y^{'} = 3x^2 y.$$

Such an equation involving a function and its derivatives is called **differential equation**. We know that the function $y = e^{x^3}$ is a solution of the above DE (differential equation).

Now, suppose that you are given the above DE:

$$\frac{dy}{dx} = 3x^2 y\,,$$

having no idea how it has been constructed, and you are asked to **find all function** $y(x)$ **satisfying this DE**.

In other words you are asked: Solve the DE

$$\frac{dy}{dx} = 3x^2 y\,,$$

which is one of the basic problems in a Course on Differential Equations. We shall study Methods to solve Differential Equations.

Solve the given Differential Equation. (Find all functions $y(x)$ satisfying the given DE.)

$$y^{'} = 3x^2y.$$

Solution by using the method of separation of the variables. The DE is given by using Newton's prime notation of a derivative.
First we revrite the given DE by using the Leibnitz's notation of a derivative:

$$\frac{dy}{dx} = 3x^2y\,.$$

Next, we separate the variables y and x and after, we integrate both sides:

$$\frac{dy}{y} = 3x^2dx \quad \Rightarrow \int \frac{dy}{y} = \int 3x^2dx$$

to obtain

$$\ln|y| + c_1 = x^3 + c_2 \quad \Rightarrow \quad \ln|y| = x^3 + c_2 - c_1$$

and if we substitute $c = c_2 - c_1$ we obtain

$$\ln|y| = x^3 + c \quad \Rightarrow \quad e^{\ln|y|} = e^{x^3+c} \quad \Rightarrow \quad |y| = e^c e^{x^3}$$

and replacing e^c by $c > 0$ we obtain

$$|y| = c\,e^{x^3},\ c > 0$$

and after removing the condition $c > 0$ we obtain the solution

$$y = c\,e^{x^3},\ c \text{ is an arbitrary constant}$$

This is a one-parameter family of functions, depending on one arbitrary constant c, representing the entire set of solutions of the given DE. The particular solution for $c = 1$ gives the function that we have started our considerations with.

The problem to solve a given DE can be considered as a far going extension (generalization) of the inverse problem of Calculus: Given a derivative, find the set of all anti-derivatives is the most simple differential equation:
Example. Given

$$y^{'}(x) = \frac{1}{1+x^2}\,.$$

Find $y(x)$. **Another formulation: Solve the given DE.**

Differential Equation (DE). An equation containing the derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation (DE).
In order to study and solve differential equations (DEs) we classify them by **Type, Order, and Linearity.**

Classification of DEs by Type.

Ordinary Differential Equations (ODE). If a DE contains only ordinary derivatives on one or more dependent variables **with respect to only one, a single independent variable**, it is called **an ordinary DE (ODE)**.

Ordinary derivatives are denoted either by using Leibnitz's notation

$$\frac{dy}{dx}; \quad \frac{d^2y}{dx^2}; \quad \frac{d^3y}{dx^3}; \quad \ldots$$

or by using Newton's prime (arabic number also) notation

$$y^{'}; \quad y^{''}; \quad y^{(3)}, y^{'''}; \quad y^{(4)}; \quad y^{(5)}; \quad \ldots$$

Examples. The DEs

$$\frac{dy}{dx} - 3x^2y = 0 \quad \text{or by prime notations} \quad y^{'} - 3x^2y = 0$$

$$\frac{d^2y}{dx^2} + 4y = e^x \quad \text{or by prime notations} \quad y^{''} + 4y = e^x$$

$$\frac{dx}{dt} + \frac{dy}{dt} = tx^2 - \sin(t)y \quad \text{or by prime notations} \quad x^{'} + y^{'} = tx^2 - \sin(t)y$$

are ODEs.

Partial Differential Equations (PDE). An equation involving the partial derivatives of one or more dependent variables with respect to **two or more independent variables** is called **partial differential equation (PDE)**.

Examples. With $u = u(x, t)$ the differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$$

is a PDE with dependent variable u and two independent variables x and t.
With $v = v(y, x)$ the differential equation

$$\frac{\partial v}{\partial y} + \frac{\partial v}{\partial x} = 0$$

is a PDE with one dependent variable v and two independent variables y and x.

Examples of ODEs (only one independent variable).

Leibnitz Notation		Prime Notation
$\frac{dy}{dx} + 5xy = e^{x^2}$	$\Leftrightarrow$	$y' + 5xy = e^{x^2}$
$\frac{d^2y}{dx^2} - \frac{1}{1+x^2}\frac{dy}{dx} + e^{3x}y = \sin(x)$	$\Leftrightarrow$	$y'' - \frac{1}{1+x^2}y' + e^{3x}y = \sin(x)$
$x^2\frac{d^3y}{dx^3} + 5e^x\frac{d^2y}{dx^2} - e^{-x^2}y = \ln(x)$	$\Leftrightarrow$	$x^2y^{(3)} + 5e^xy'' - e^{-x^2}y = \ln(x)$
$t^2\frac{dx}{dt} + 5\cos(t)\frac{d^2y}{dt^2} = e^tx + y - \sqrt{t}$	$\Leftrightarrow$	$t^2x' + 5\cos(t)y'' = e^tx + y - \sqrt{t}$

Remark. Although less convenient to write, the Leibnitz notation has an advantage over the Newton's prime notation in clearly displaying both dependent and independent variables. For example, in the DE

$$\frac{d^2x}{dt^2} - x = 0$$

it is seen that the symbol x denotes the dependent variable and the independent variable is denoted by t. The corresponding prime notation $x'' - x = 0$ does not give such detailed information but it is convenient in the process of solving DEs.

Remark. In physical science and engineering Newton's dot notation is sometimes used to denote derivatives with respect to the time t. For example the DE: $\frac{d^2s}{dt^2} = -32$ becomes $\ddot{s} = -32$ by using dot notations.

In this course we shall study Ordinary Differential Equations (ODEs), i.e., Differential Equations with One Independent Variable. So, we continue our considerations with ODEs.

Classification of DEs by Order. Order of DE. The order of the highest derivative in a DE is called **order of the DE**.
Example. The DE

$$x^5\frac{d^2y}{dx^2}+\frac{1}{1+x^3}\left(\frac{dy}{dx}\right)^3+3y^4=xe^x \quad \Leftrightarrow \quad x^5y^{''}+\frac{1}{1+x^3}\left(y^{'}\right)^3+3y^4=xe^x$$

is a second-order DE with only one independent variable x hence, it is a second-order, ordinary differential equation (ODE).
Differential Form of a First-Order DE. First-order ODE having only one dependent variable (and one independent variable because it is ordinary), can be written in a form

$$M(x,y)dx+N(x,y)dy=0$$

called **a differential form**.

Example. Consider the first-order DE given in **a differential form**:

$$(y^2+xy)dx+x^2dy=0.$$

Assume that y is the dependent variable ($y=y(x)$) in the given DE. Then it can be written as

$$x^2\frac{dy}{dx}+y^2+xy=0 \quad \Leftrightarrow \quad x^2y^{'}+y^2+xy=0.$$

Assume that x is the dependent variable ($x=x(y)$) in the given DE. Then it can be written also as

$$(y^2+xy)\frac{dx}{dy}+x^2=0=0 \quad \Leftrightarrow \quad (y^2+xy)x^{'}+x^2=0.$$

This example shows how useful is the Leibnitz notation for derivatives.

Normal Form of a DE. The n-th order ODE with only one dependent variable y and only one independent variable x has the general form

$$F(x, y, y^{'}, \ldots, y^{(n)}) = 0,$$

where F is a function of $n+2$ variables. Solving the above equation with respect to the highest derivative we obtain the so called **normal form of a DE**:

$$y^{(n)} = f(x, y^{'}, \ldots, y^{(n-1)}).$$

Examples. (a) In the above example, assuming y dependent variable, the normal form of the DE $\quad (y^2 + xy)dx + x^2 dy = 0$ is:

$$y^{'} = -\frac{y^2 + xy}{x^2}.$$

(b) The normal form of the DE $\quad 3xy\frac{dy}{dx} + 5e^x y = x^2 + 1$ is

$$\frac{dy}{dx} = \frac{x^2 - 5e^x y + 1}{3xy}.$$

(c) The normal form of the DE

$$e^{x+y}\frac{d^2y}{dx^2} - e^{3x-y}\frac{dy}{dx} + e^{x+y^2}y - x^2 = 0$$

is

$$\frac{d^2y}{dx^2} = e^{2x-2y}\frac{dy}{dx} - e^{y^2-y}y + e^{-x-y}x^2.$$

Classification by Linearity. An n-th order **linear ODE** has the form:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$\Leftrightarrow a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y^{'} + a_0(x)y = g(x),$$

where
the dependent variable y and its derivatives $y^{'}, y^{''}, \ldots, y^{(n)}$ are of power 1 and the coefficients $a_0(x), a_1(x), \ldots, a_n(x)$ depend only on the independent variable x but not on the dependent variable y.

General Form of first-order, linear ODE:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad \Leftrightarrow \quad a_1(x)y^{'} + a_0(x)y = g(x).$$

General Form of second-order, linear DE:

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad \Leftrightarrow \quad a_2(x)y^{''} + a_1(x)y^{'} + a_0(x)y = g(x).$$

General Form of third-order, linear DE:

$$a_3(x)\frac{d^3y}{dx^3} + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$\Leftrightarrow a_3(x)y^{(3)} + a_2(x)y^{''} + a_1(x)y^{'} + a_0(x)y = g(x).$$

Remark. Note, the linearity is a property of a given DE concerning the dependent variable.

Problem 1. State the order of the given DE. Determine whether it is linear or nonlinear.

$$\textbf{(a)} \quad (1-x^2)y^{''} - \frac{4}{1+x^3}y^{'} + \cos(x)y = \tan(x).$$

The given DE is second-order, linear.

$$\textbf{(b)} \quad (x^2+1)\frac{dy}{dx} - \cos(x)y = 0.$$

The given DE is first-order, linear.

$$\textbf{(c)}\quad \frac{d^2u}{dt^2} + \frac{du}{dt} + \sqrt{u} = e^t.$$

The given DE is second-order, non-linear because of the non-linear term $\sqrt{u}$.

$$\textbf{(d)}\quad y^{(3)} - x^2(y^{'})^2 + e^x y = \frac{\ln(x)}{x^4+2}.$$

The given DE is third-order, non-linear because of the non-linear term $(y^{'})^2$.

$$\textbf{(e)}\quad \ddot{x} + (1+\dot{x})\dot{x} + x = v^2.$$

The given DE is in Newton's dot notations. It is second-order, non-linear because of the non-linear term $(\dot{x})^2$. Note that $(1+\dot{x})\dot{x} = \dot{x} + (\dot{x})^2$.

$$\textbf{(f)}\quad e^{x+y}\frac{dy}{dx} - x^2 y = 3x^2 - x.$$

The given DE is first-order, non-linear because of the non-linear term e^y in y. Note that $e^{x+y} = e^x\, e^y$.

$$\textbf{(g)}\quad t^2\frac{d^3x}{dt^3} - e^{t^2+1}\frac{dx}{dt} + (\ln(t)+1)x = \frac{\cos(t)}{t+3}.$$

The given DE is third-order, linear.

$$\textbf{(h)}\quad y\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^2 + (x+1)\frac{dy}{dx} - \frac{1}{x^2}y = 0.$$

The given DE is third-order, non-linear because of the non-linear terms $y\frac{d^3y}{dx^3}$ and $\left(\frac{d^2y}{dx^2}\right)^2$.

$$\textbf{(i)}\quad y\frac{dy}{dx} + x(1+y^2) = 0.$$

The given DE is first-order, non-linear because of the non-linear terms $y\frac{dy}{dx}$ and y^2.

$$\textbf{(j)}\quad e^{-x^2}y^{(4)} + 2\sin(x)y^{''} - \frac{1}{y} = \frac{1}{x}.$$

The given DE is fourth-order, non-linear because of the non-linear term $\frac{1}{y} = y^{-1}$.

$$\textbf{(k)}\quad (1+x^3)y^{''} + \sin(x)y = \cos(x+y).$$

The given DE is second-order, non-linear because of the non-linear term $\cos(x+y)$ in y.

$$\textbf{(l)} \quad (1+x^3)y^{''} + \sin(x)y = \cos(x).$$

The given DE is second-order, linear.

$$\textbf{(m)} \quad x^2 y dx + (1+x)dy = 0.$$

The given DE is in differential form. Assuming that y is the dependent variable, then it can be written as

$$(1+x)\frac{dy}{dx} + x^2 y = 0$$

and we conclude that the given DE is first-order, linear in y.
Assuming that x is the dependent variable it can be written as

$$x^2 y \frac{dx}{dy} + 1 + x = 0.$$

Therefore, the given DE is first-order, non-linear in x because of the non-linear term $x^2 \frac{dx}{dy}$. Hence, a first-order ODE can be linear in one of its variables and non-linear with respect to the other of its variables. Next problem is an example.

Problem 2. The first-order DE is given in differential form. Determine whether the given first-order DE is linear in the indicated dependent variable.

$$\textbf{(a)} \quad (x^2+1)dy \;-\; \cos(x)y\,dx \;=\; 0.$$
$$\textbf{(b)} \quad (x^3-1)\,dy \;+\; y\,dx \;=\; 0 \quad \text{in } y; \quad \text{in } x.$$
$$\textbf{(c)} \quad (u^2+1)\,dv + (v - u^2 v)\,du \;=\; 0 \quad \text{in } v; \quad \text{in } u.$$

Solution. $\cdots$

Problem 3. Verify if the given function $y = f(x)$ is a solution of the given DE. Give at least one interval I where the solution $y = f(x)$ is well defined.

$$\textbf{(a)} \quad y^{'} = 2xy^2, \quad y = \frac{1}{4-x^2}.$$

Solution. We have

$$y^{'} - 2xy^2 = 0, \quad f(x) = (4-x^2)^{-1}, f^{'}(x) = (-1)(4-x^2)^{-2}(-2x) = 2x(4-x^2)^{-2}.$$

Then

$$y' - 2xy^2 = 2x(4-x^2)^{-2} - (2x)[(4-x^2)^{-1}]^2 = 2x(4-x^2)^{-2} - 2x(4-x^2)^{-2} = 0\,.$$

Hence, $y = \frac{1}{4-x^2}$ is a solution of the given DE on any interval I, where $y = \frac{1}{4-x^2}$ is differentiable.
The function $y = \frac{1}{4-x^2}$ is differentiable for all x such that $4 - x^2 \neq 0 \Leftrightarrow x \neq \pm 2$. Hence, $I = (-\infty, -2)$, $I = (-2, 2)$, $I = (2, \infty)$ are intervals, where the given DE has a solution $y = \frac{1}{4-x^2}$.

$$\textbf{(b)} \quad y' = 25 + y^2, \quad y = 5\tan(5x).$$

Solution. We have $f(x) = 5\tan(5x)$. Representing the given DE in an equivalent form and computing the derivative of $f(x)$ we obtain:

$$y' - y^2 - 25 = 0, \quad y(x) = 5\tan(5x), y'(x) = 5\frac{1}{\cos^2(5x)}[5x]' = \frac{25}{\cos^2(5x)}.$$

Then, replacing y and y' in the given d.e. we obtain:

$$\begin{aligned} y' - y^2 - 25 &= \frac{25}{\cos^2(5x)} - [5\tan(5x)]^2 - 25 \\ &= \left(\frac{25}{\cos^2(5x)} - 25\right) - 25\tan^2(5x) \\ &= 25\frac{1-\cos^2(5x)}{\cos^2(5x)} - 25\tan^2(5x) \\ &= 25\frac{\sin^2(5x)}{\cos^2(5x)} - 25\tan^2(5x) \\ &= 25\tan^2(5x) - 25\tan^2(5x) = 0\,. \end{aligned}$$

We conclude that $y = 5\tan(5x)$ is a solution of the given DE on any interval I, where $y = 5\tan(5x)$ is differentiable. In the above computations we use the trigonometric identity $1 - \cos^2(5x) = \sin^2(5x)$.
The function $y = 5\tan(5x)$ is differentiable for

$$-\pi/2 < 5x < \pi/2 \quad \Leftrightarrow \quad -\pi/10 < x < \pi/10.$$

Hence, $I = (-\pi/10, \pi/10)$ is an interval, where the given DE has a solution $y = 5\tan(5x)$.

Trivial Solution. The DE $y' = 2xy^2$ in the above problem has the constant solution $y = 0$ on $I = (-\infty, \infty)$. A solution of a DE on an interval I that is identically zero is called **a trivial solution.**

Problem 4. Verify if the function $y = e^{3x}\cos(2x)$ is a solution of the DE

$$y'' - 6y' + 13y = 0$$

on the interval $I = (-\infty, \infty)$.
Solution. We compute with $f(x) = e^{3x}\cos(2x)$

$$\begin{aligned} f(x) &= e^{3x}\cos(2x) \\ f'(x) &= 3e^{3x}\cos(2x) - 2e^{3x}\sin(2x) \\ f''(x) &= 9e^{3x}\cos(2x) - 12e^{3x}\sin(2x) - 4e^{3x}\cos(2x) \\ &= 5e^{3x}\cos(2x) - 12e^{3x}\sin(2x) \end{aligned}$$

Then,

$$\begin{aligned} &y'' - 6y' + 13y = f''(x) - 6f'(x) + 13f(x) \\ &[5e^{3x}\cos(2x) - 12e^{3x}\sin(2x)] - 6[3e^{3x}\cos(2x) - 2e^{3x}\sin(2x)] \\ &+13e^{3x}\cos(2x) \\ &= (5 - 18 + 13)e^{3x}\cos(2x) + (12 - 12)e^{3x}\sin(2x) = 0. \end{aligned}$$

Taking into account that $f(x) = e^{3x}\cos(2x)$ has a second continuous derivative on $(-\infty, \infty)$, we conclude that $f(x) = e^{3x}\cos(2x)$ is a solution of the given DE on the interval $(-\infty, \infty)$.

Problem 5. Given the one-parameter family function

$$P = \frac{ce^t}{1 + ce^t},$$

where c is an arbitrary constant. Verify that the each function P of the given family is a solution of the DE:

$$\frac{dP}{dt} = P(1 - P).$$

Solution. Applying the quotient rule for differentiation we obtain

$$\frac{dP}{dt} = \frac{ce^t(1+ce^t) - (ce^t)(ce^t)}{(1+ce^t)^2} = \frac{ce^t}{(1+ce^t)^2}.$$

and representing the given DE in the form

$$\frac{dP}{dt} - P(1-P) = 0$$

we compute

$$\begin{aligned}
\frac{dP}{dt} - P(1-P) &= \frac{ce^t}{(1+ce^t)^2} - \frac{ce^t}{1+ce^t}\left(1 - \frac{ce^t}{1+ce^t}\right) \\
&= \frac{ce^t}{(1+ce^t)^2} - \frac{ce^t}{1+ce^t}\left(\frac{1+ce^t-ce^t}{1+ce^t}\right) \\
&= \frac{ce^t}{(1+ce^t)^2} - \frac{ce^t}{1+ce^t}\left(\frac{1}{1+ce^t}\right) \\
&= \frac{ce^t}{(1+ce^t)^2} - \frac{ce^t}{(1+ce^t)^2} = 0\,.
\end{aligned}$$

We conclude that the function P is a solution of the given DE for any arbitrarily chosen constant c on an interval I such that $1 + ce^t \neq 0$. For example, if $c > 0$, then $I = (-\infty, \infty)$.

Problem 6. Find the values of m such that the function $y = e^{mx}$ is a solution of the given DE

(a) $y' + 2y = 0$;
(b) $y'' - 5y' + 6y = 0$.

Solution. **(a)** We compute $y = e^{mx}$, $y' = me^{mx}$ and in order $y = e^{mx}$ to be a solution of the given DE we must have:

$$y' + 2y = me^{mx} + 2e^{mx} = (m+2)e^{mx} = 0.$$

Taking into account that $e^{mx} > 0$ for all m and x we conclude that m has to satisfy the linear equation $m + 2 = 0$. Hence, $m = -2$. From here, $y = e^{-2x}$ is the unique function of the form $y = e^{mx}$ that is a solution of the given DE.
(b) We compute $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2e^{mx}$ and in order $y = e^{mx}$ to be a solution of the given DE we must have:

$$y'' - 5y' + 6y = m^2e^{mx} - 5me^{mx} + 6e^{mx} = (m^2 - 5m + 6)e^{mx} = 0.$$

Taking into account that $e^{mx} > 0$ for all m and x we conclude that m must be solution of the quadratic equation $m^2 - 5m + 6 = 0$ that is $(m-2)(m-3) = 0$. Hence, $m = 2$ and $m = 3$. From here, the functions $y = e^{2x}$ and $y = e^{3x}$ are the only functions of the form $y = e^{mx}$ that are solutions of the given DE.

Problem 7. **(a)** Verify that the one-parameter family of functions

$$y = \frac{1}{x^2 + c}$$

are solutions of the DE

$$y' + 2xy^2 = 0.$$

(b) Find a solution of the given DE satisfying the initial value condition $y(-3) = 1/3$. Give the largest interval over which the solution obtained is well defined.

Solution. **(a)** We compute

$$y = \frac{1}{x^2 + c}, \quad y' = \frac{-2x}{(x^2 + c)^2}$$

and

$$\begin{aligned} y' + 2xy^2 &= \frac{-2x}{(x^2+c)^2} + (2x)\left(\frac{1}{x^2+c}\right)^2 \\ &= \frac{-2x}{(x^2+c)^2} + \frac{2x}{(x^2+c)^2} = 0. \end{aligned}$$

Hence, for an arbitrary constant c, the function $y = \frac{1}{x^2+c}$ is a solution of the given DE.
(b) We are looking for a function of the form

$$y(x) = \frac{1}{x^2 + c}$$

such that $y(-3) = 1/3$ that is

$$y(-3) = \frac{1}{(-3)^2 + c} = \frac{1}{3} \quad \Leftrightarrow \quad \frac{1}{9+c} = \frac{1}{3} \quad \Leftrightarrow \quad 9 + c = 3 \Rightarrow c = -6.$$

Hence, the function

$$y = \frac{1}{x^2 - 6}$$

is a solution of the given DE satisfying the initial-value condition $y(-3) = 1/3$. Obviously, this solution is well defined for all x such that $x^2 - 6 \neq 0 \Leftrightarrow x \neq \pm\sqrt{6}$. Hence, $I = (-\infty, -\sqrt{6})$ or $I = (\sqrt{6}, \infty)$. Also, note that the solution is well defined on the interval $I = (-\sqrt{6}, \sqrt{6})$. Evidently, this interval is of smaller length.

Problem 8. Consider the DE

$$\frac{dy}{dx} = y(a - by),$$

where $a > 0$ and $b > 0$ are positive constants. Find two constant solutions of the given DE. Find intervals on the y-axis on which a non-constant solution of the given DE is either increasing or decreasing.

Solution. Suppose that $y = c$ is a constant solution. Then, taking into account that $y^{'} = 0$ we must have

$$c(a - bc) = 0 \quad \Rightarrow \quad c = 0 \text{ or } a - bc = 0.$$

In view of this $y = 0$ and $y = \frac{a}{b}$ are two constant solutions of the given DE.
Now suppose that a non-constant solution y is in the interval $(-\infty, 0)$, i.e., $y < 0$. Then $y^{'} = y(a - by) < 0$, in other words the derivative of the solution y will be negative by sign. Hence, the solution y will be decreasing.
Suppose that a non-constant solution is in the interval $(0, a/b)$, i.e., $0 < y < a/b$. Then, $y^{'} = y(a - by) > 0$ and from here the solution y, having a positive derivative, will be increasing.
Suppose that a non-constant solution y is in the interval $(a/b, \infty)$, i.e., $y > a/b$. Then, $y^{'} = y(a - by) < 0$ and from here the solution y, having a negative derivative, derivative, will be decreasing.

Remark. In the next section we shall consider Picard's Existence and Uniqueness Theorem (Theorem 1.2.1/page 15, textbook)
http://mathworld.wolfram.com/PicardsExistenceTheorem.html
In view of this Theorem if for a fixed x_0, the functional value $y(x_0)$ belongs to the interval $(-\infty, 0)$, then the solution $y(x)$ never leaves this interval. Analogously, if for a fixed x_0, the functional value $y(x_0)$ belongs to the interval $(0, a/b)$, then the solution $y(x)$ never leaves this interval. Similarly, if for a fixed x_0, the functional value $y(x_0)$ belongs to the interval $(a/b, \infty)$, then the solution $y(x)$ never leaves this interval.

Explicit Solution. A solution in which the dependent variable is expressed only in terms of the independent variable and some constants is said to be **an explicit solution.**

Example. The DE $y'' - 2y' + y = 0$ has an explicit **2-parameter family of solutions** $y(x) = c_1 e^x + c_2 x e^x$, where c_1 and c_2 are arbitrary constants. Verify this fact.

We compute:

$$y' = c_1 e^x + c_2 e^x + c_2 x e^x; \quad y'' = c_1 e^x + 2c_2 e^x + c_2 x e^x$$

and

$$\begin{aligned}&y'' - 2y' + y = (c_1 e^x + 2c_2 e^x + c_2 x e^x) - 2\left(c_1 e^x + c_2 e^x + c_2 x e^x\right) + (c_1 e^x + c_2 x e^x)\\&= (c_1 - 2c_1 + c_1)e^x + (2c_2 - 2c_2)e^x + (c_2 - 2c_2 + c_2)xe^x = 0.\end{aligned}$$

Implicit Solution. However, some of the methods of solving DEs do not lead directly to an explicit solution. Often we have to be content with a relation of the form $G(x, y) = 0$ that defines a solution $y = f(x)$ implicitly.

Definition. A relation $G(x, y) = 0$ is said to be an implicit solution of a given DE on an interval I if there exists at least one function $y = f(x)$ that satisfies this relation as well as the differential equation on I.
It is not in the scope of this course to study conditions under which a relation $G(x, y) = 0$ defines a differentiable function $y = f(x)$. We shall assume in this course the following: If a formal application of a method to solve a DE leads to a relation $G(x, y) = 0$, then there exists at least one function $y = f(x)$ satisfying this relation and the DE.
However, note that if the implicit solution $G(x, y) = 0$ is fairly simple it may be possible to solve it for y and to obtain one or more explicit solutions.

Example. The relation $x^2 + y^2 = 4$ is an implicit solution of the DE $\frac{dy}{dx} = -\frac{x}{y}$. This implicit solution gives two explicit solutions when solving the relation $x^2 + y^2 = 4$ with respect to y:

$$y_1(x) = \sqrt{4 - x^2} \;\; x \in (-2, 2); \quad y_2(x) = -\sqrt{4 - x^2} \;\; x \in (-2, 2).$$

Note that we take the open interval $(-2, 2)$ because at -2 and at 2 both $y_1(x)$ and $y_2(x)$ are not differentiable. Hence, the graph of the implicit solution $x^2 + y^2 = 4$ is the circle of radius 2 centered at the origin but $y_1(x) = \sqrt{4-x^2}$ and $y_2(x) = -\sqrt{4-x^2}$ are two distinct explicit solutions.

Family of Solutions and Particular Solution. When solving a first-order DE $F(x, y, y^{'}) = 0$ we usually obtain a solution containing an arbitrary constant c, in other words, we obtain infinitely many solutions depending on a constant c, i.e., **an one-parameter family of solutions.**

Example. The first-order linear DE $xy^{'} - y = x^2 \sin(x)$ has **the one-parameter family of solutions** $y = cx - x\cos(x)$ on $(-\infty, \infty)$, where c is an arbitrary constant. Now with the particular choice of c: $c = 0$ we obtain **the particular solution** $y = -x\cos(x)$.
When solving a second-order DE $F(x, y, y^{'}, y^{''}) = 0$ we usually obtain a solution containing two arbitrary constant c_1 and c_2, in other words, we obtain infinitely many solutions depending on two constants c_1 and c_2, i.e., **a two-parameter family of solutions.**

Example. The second order DE $y^{''} - 5y^{'} + 6y = 0$ has **the two-parameter family of solutions** $y = c_1e^{2x} + c_2e^{3x}$, where c_1 and c_2 are arbitrary constants. With the particular choice $c_1 = 1$ and $c_2 = -1$ we obtain **the particular solution** $y = e^{2x} - e^{3x}$ of the given DE.

1.2. Initial Value Problem (IVP)

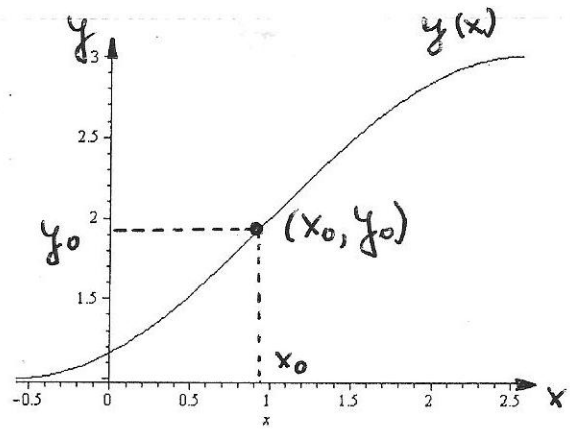
Formulation of an n-order IVP. Let n be a positive integer ($n = 1, 2, 3, \ldots$). Given a number x_0 and an interval I containing x_0, i.e., $x_0 \in I$.
The problem:

$$\begin{aligned} \text{Solve:} \quad & y^{(n)} = f\left(x, y, y', \ldots, y^{(n-1)}\right) \\ \text{Subject to:} \quad & y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1} \end{aligned}$$

for given numbers $y_0, y_1, \ldots, y_{n-1}$ is called **n-order initial-value problem (n-order IVP)**.
Note that in the above formulation, the n-order DE is given in a normal form. Here are some particular cases.
First-order IVP:

$$y' = f(x, y), \quad \text{s.t.} \quad y(x_0) = y_0.$$

Geometrically, we are looking for a solution $y = f(x)$ of the given first-order DE passing through the point with coordinates (x_0, y_0), i.e., $y(x_0) = y_0$:



First-order IVP.

Problem 1. Verify that $y = e^{x^2/2}$ is a solution of the first-order IVP:

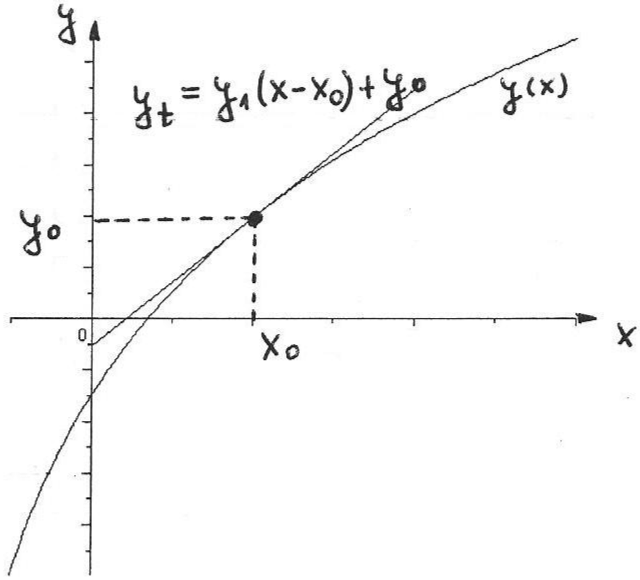
$$y' = xy, \quad \text{s.t.} \quad y(0) = 1.$$

Solution. First, we verify that $y = e^{x^2/2}$ is a solution of the DE: $y' = xe^{x^2/2} = xy$. Next, $y(0) = e^0 = 1$ hence, $y = e^{x^2/2}$ satisfies the initial-value condition, also.

Second-order IVP:

$$y'' = f(x, y, y'), \quad \text{s.t.} \quad y(x_0) = y_0,\ y'(x_0) = y_1.$$

Geometrically, we are looking for a solution $y = f(x)$ of the given second-order DE passing through the point with coordinates (x_0, y_0), i.e., $y(x_0) = y_0$, such that the slope of the tangent line $y_t = y_1(x - x_0) + y_0$ to the graph of $y = y(x)$ at the point (x_0, y_0) is equal to y_1, i.e., $y'(x_0) = y_1$:



Second-order IVP.

Problem 2. Verify that $y = 2e^x - x^2/2 - x - 2$ is a solution of the second-order IVP:

$$y'' = y' + x, \quad \text{s.t.} \quad y(0) = 0,\ y'(0) = 1.$$

Solution. First, we verify that the given function is a solution of the DE:

$$y' + x = (2e^x - x - 1) + x = 2e^x - 1, \quad y'' = 2e^x - 1$$

hence, $y'' = y' + x$. Also, $y(0) = 2e^0 - 2 = 2 - 2 = 0$, $y'(0) = 2^0 - 1 = 2 - 1 = 1$. We conclude that $y = 2e^x - x^2/2 - x - 2$ is a solution of the given DE satisfying the given initial-value conditions hence, $y = 2e^x - x^2/2 - x - 2$ is a solution of the given second-order IVP.

Here is **an important Theorem due to Picard http://mathworld.wolfram.com/PicardsExistenceTheorem.html on the existence and uniqueness of a solution of a first-order IVP.**

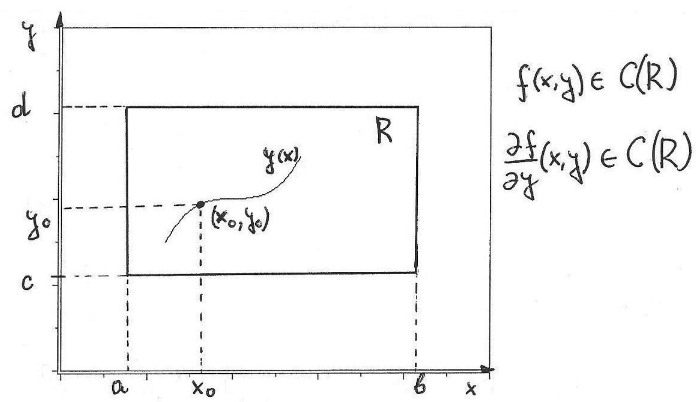
Theorem 1 (Picard's Existence and Uniqueness Theorem 1.2.1/ page 15, textbook). Let R be a rectangular region in the (x, y)-plane, defined by $a \le x \le b$ and $c \le y \le d$, i.e.,

$$R = \{(x, y) \ : \ a \le x \le b, \ c \le y \le d\} .$$

Let the point (x_0, y_0) be in the interior of R, i.e., $a < x_0 < b, \ c < y_0 < d$.
Consider the first-order IVP:

$$y' = f(x, y), \quad \text{s.t.} \quad y(x_0) = y_0.$$

If the two-variable functions $f(x, y)$ and its first partial derivative $\frac{\partial f}{\partial y}(x, y)$ with respect to y are continuous on R, then:
There exists an interval $I_0 = \{x \ : \ x_0 - h < x < x_0 + h, \ h > 0\}$ **contained in the interval** $a \le x \le b$ **and a unique function** $y = f(x)$**, defined on the interval** I_0 **that is a solution of the given first-order IVP.**
In other words, the given first-order IVP has a unique solution on the interval $(x_0 - h, x_0 + h)$ **for some** $h > 0$**.**



Picard's Theorem.

Problem 3. Use Picard's Theorem 1 in order to show that the first-order IVP

$$y' = x^3y^2, \quad y(0) = 1$$

has a unique solution in $(-h, h)$ for some $h > 0$.

Solution. The given DE has the form $y' = f(x, y)$, where $f(x, y) = x^3y^2$. The functions

$$f(x, y) = x^3y^2, \quad \frac{\partial f}{\partial y}(x, y) = 2x^3y$$

are continuous on the rectangle $R = [-2, 2] \times [0, 2]$

$$R = \{(x, y) \;:\; -2 \leq x \leq 2, \; 0 \leq y \leq 2\}$$

containing the point $(0, 1)$. **Hence, according to Theorem 1 there is only one solution $\mathbf{y(x)}$ of the given DE passing through the points $(0, 1)$ in an interval** $x \in (-h, h)$, $h > 0$, (having a midpoint at 0). **In other words, there is an $\mathbf{h > 0}$ such that the given IVP has a unique solution $\mathbf{y(x)}$ in $\mathbf{x \in (-h + 0, 0 + h) = (-h, h)}$.**

Let us find explicitly the unique solution of the given above IVP by solving the given IVP. We separate the variables and integrate to solve the DE, in other words we apply the method of separation of the variables:

$$\begin{aligned}
&\frac{dy}{y^2} = x^3dx \quad \Rightarrow \quad \int \frac{dy}{y^2} = \int x^3dx \\
&-y^{-1} + c_1 = \frac{x^4}{4} + c_2 \quad \Rightarrow \quad -y^{-1} = \frac{x^4}{4} + c \\
&\frac{1}{y} = c - \frac{x^4}{4} \quad \Rightarrow \quad y = \frac{4}{c - x^4}.
\end{aligned}$$

We conclude that the given first-order DE has a one-parameter family of solutions. Applying the initial-value condition will determine a unique value of the parameter c and from here, the unique solution of the given first-order IVP:

$$y(0) = 1 \quad \Rightarrow \quad y(0) = \frac{4}{c - 0^4} = \frac{4}{c} \quad \Rightarrow \quad \frac{4}{c} = 1 \quad \Rightarrow \quad c = 4.$$

Hence, the unique solution of the given IVP is

$$y = \frac{4}{4 - x^4}, \quad -\sqrt{2} < x < \sqrt{2}, \quad h = \sqrt{2}$$

and **this confirms the theoretical conclusion made above by using Theorem 1.**

Example. Consider the first-order IVP

$$y' = x\sqrt{y}, \quad y(1) = 0.$$

We have

$$f(x,y) = x\sqrt{y}, \quad \frac{\partial f}{\partial y} = x\frac{1}{2}y^{1/2-1} = \frac{x}{2}y^{-1/2} = \frac{x}{2\sqrt{y}}.$$

Hence, the 2-variable function $\partial f/\partial y$ is not continuous at $(1,0)$ and from here it is not continuous on any rectangle containing this point.
Hence, Theorem 1 is not applicable on such a rectangle because the conditions given in Theorem 1 are not satisfied. From here, we can not conclude by Theorem 1 that the given IVP has a unique solution.

In fact, the given IVP has more than one solution. We show by solving the given IVP that the given IVP has at least 2 explicit solutions:
First, the constant function $y(x) = 0 \Rightarrow y'(x) = 0$ is a solution of the given IVP.
Second, we solve the IVP by separating the variables:

$$\frac{dy}{dx} = xy^{1/2} \quad \Rightarrow \quad \frac{dy}{y^{1/2}} = xdx$$

$$\Rightarrow \int y^{-1/2}dy = \int xdx \quad \Rightarrow \quad \frac{y^{1/2}}{1/2} = \frac{x^2}{2} + c$$

$$\Rightarrow 2y^{1/2} = \frac{x^2}{2} + c \quad \Rightarrow y^{1/2} = \frac{x^2}{4} + c$$

$$\Rightarrow y = \left(\frac{x^2}{4} + c\right)^2, \quad 0 = y(1) = \left(\frac{1}{4} + c\right)^2 \quad \Rightarrow \quad c = -\frac{1}{4}$$

$$y = \left(\frac{x^2}{4} - \frac{1}{4}\right)^2.$$

Hence, the given IVP has at least 2 solutions:

$$y_1(x) = 0, \quad y_2(x) = \left(\frac{x^2}{4} - \frac{1}{4}\right)^2.$$

Problem 4. The two-parameter family of functions

$$y = c_1 e^x + c_2 e^{-x}, \quad c_1 \text{ and } c_2 \text{ arbitrary constants}$$

is a solution of the linear, second-order DE:

$$y^{''} - y = 0.$$

Find a solution of the second order IVP consisting of the given DE and the given initial-value conditions:

$$\textbf{(a)} \quad y(0) = 1, \quad y^{'}(0) = 2.$$

Solution.

$$y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 = 1.$$
$$y^{'}(x) = c_1 e^x - c_2 e^{-x} \quad \Rightarrow \quad y^{'}(0) = c_1 - c_2 = 2.$$

Hence, in order to find c_1 and c_2 we have to solve the linear system:

$$c_1 + c_2 = 1$$
$$c_1 - c_2 = 2$$

We sum the two equation to obtain $c_1 = 3/2$ and from here $c_2 = 1 - c_1 = -1/2$. The unique solution of the second-order IVP is

$$y = \frac{3}{2}e^x - \frac{1}{2}e^{-x}.$$

$$\textbf{(b)} \quad y(1) = 0, \quad y^{'}(1) = e.$$

Solution. We compute

$$y(1) = c_1 e + c_2 e^{-1} = 0 \quad \Rightarrow \quad c_1 e + c_2 e^{-1} = 0$$
$$y^{'}(1) = c_1 e - c_2 e^{-1} = e \quad \Rightarrow \quad c_1 e - c_2 e^{-1} = e.$$

We sum the 2 equations to obtain $2c_1 e = e \Rightarrow c_1 = 1/2$ and from here

$$c_2 e^{-1} = -c_1 e = -\frac{1}{2}e \qquad \Rightarrow c_2 = -\frac{1}{2}e^2.$$

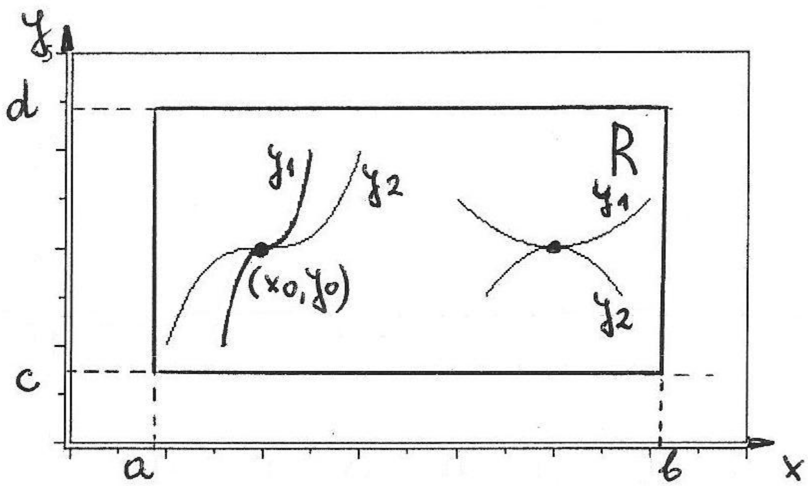
Hence, the unique solution of this second-order IVP is

$$y = \frac{1}{2}e^x - \frac{1}{2}e^{2-x}.$$

Problem 49/ page 17, textbook Suppose that the first-order DE

$$\frac{dy}{dx} = f(x,y)$$

possesses a one-parameter family of solutions and that $f(x,y)$ satisfies the conditions of Theorem 1 (Theorem 1.1/ p.17, the textbook) in some rectangular region R on the $xy-$ plane. Explain, why two different solution curves $y_1(x)$ and $y_2(x)$ of $\frac{dy}{dx} = f(x,y)$ cannot intersect each other or be tangent at an interior point (x_0, y_0) in the rectangle R.



Application of Picard's Theorem (Theorem 1).

Solution. Suppose to the contrary that there is a point (x_0, y_0) in R, where two solutions of the DE

$$\frac{dy}{dx} = f(x,y)$$

either intersects or are tangent (see the graph given above). In other words $y_1(x_0) = y_2(x_0) = y_0$. However, according to **Theorem 1**, there is an $h > 0$ such that in the interval $[x_0 - h, x_0 + h]$ there is **only one solution** of the IVP

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0.$$

From here, $y_1(x) = y_2(x)$ on $[x_0 - h, x_0 + h]$ and we get a contradiction to our assumption.

Hence, if

$$f(x,y) \quad \textbf{and} \quad \frac{\partial f}{\partial y}(x,y)$$

are continuous at any point (x, y) from R, then it is not possible for two solution curves of the DE

$$\frac{dy}{dx} = f(x,y)$$

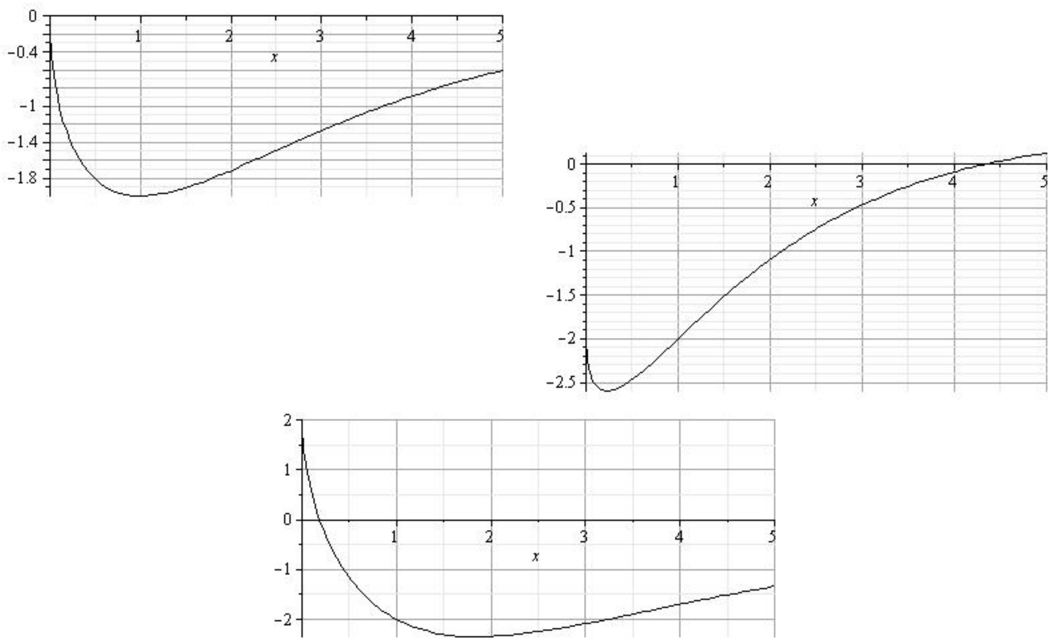
to intersect or to be tangent at an interior point (x_0, y_0) in R.

Problem 5. The graphs of 3 members of the family of solutions of a second order d.e.

$$d^2y/dx^2 = f(x, y, dy/dx) \quad \Leftrightarrow \quad y'' = f(x, y, y')$$

are given below. Match each solution curve with one pair of the following initial conditions:

(**1**) $y(1) = -2,\ y'(1) = 1;$
(**2**) $y(1) = -2,\ y'(1) = 0;$
(**3**) $y(1) = -2,\ y'(1) = -1.$

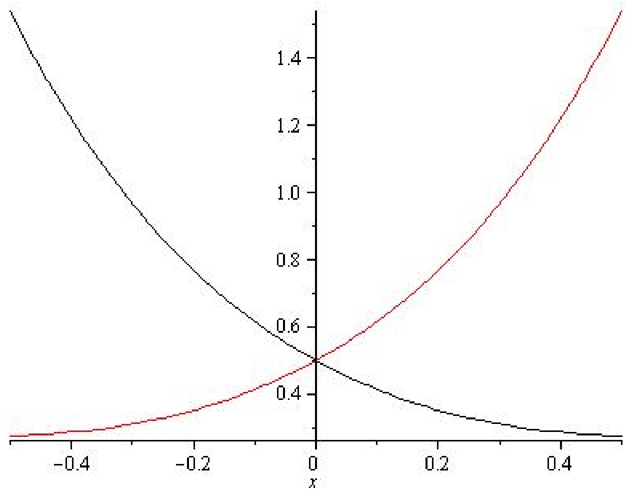


Solution curves **(a), (b), (c)**.

Problem 6. Consider the first-order IVP:

$$y' = x - 2y, \;\; y(0) = 0.5.$$

Determine which one of the two curves given below is the only possible solution curve. Hint: Compute $y'(0)$.



Problem 7. The graph of a solution of a second order IVP:

$$y'' = f(x, y, y'), \;\; y(3) = y_0, \;\; y'(3) = y_1$$

is given below. Use the graph to estimate y_0 and y_1.