

MHF4U Grade 12 Advanced Functions

Assessment of Learning 3

Task 1: Knowledge and Understanding questions

1. State the dominant term, leading coefficient, and degree of the polynomial for the function $f(x) = 3(x + 4)^2 (4x - 1)(x - 5)^2$. (3 marks)

Answer:

$$\text{Dominant term} = 3(x^2)(4x)(x^2) = 12x^5$$

$$\text{Leading coefficient} = 3 \cdot 1^2 \cdot 4 \cdot 1^2 = 12$$

$$\text{Degree} = 2 + 1 + 2 = 5$$

2. Write the following polynomial function in standard form:

$$g(x) = 4(x - 2)^3 + 5. \text{ (3 marks)}$$

Answer:

$$\begin{aligned} g(x) &= 4(x - 2)^3 + 5 \\ &= 4(x^3 - 3x^2 \cdot 2 + 3x \cdot 2^2 - 2^3) + 5 \\ &= 4(x^3 - 6x^2 + 12x - 8) + 5 \\ &= 4x^3 - 24x^2 + 48x - 32 + 5 \\ &= 4x^3 - 24x^2 + 48x - 27 \end{aligned}$$

Apply the formula:

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

3. Factor each of the following expressions completely. (8 marks: 2 marks each)

a) $4a^2b - 8a^3b^5 + 2ab^7$

Answer:

$$4a^2b - 8a^3b^5 + 2ab^7 = 2ab(2a - 4a^2b^4 + b^6)$$

b) $12x^2 - 75y^2$

Answer:

$$12x^2 - 75y^2 = 3(4x^2 - 25y^2) \\ = 3(2x - 5y)(2x + 5y)$$

Apply the formula:

$$a^2 - b^2 = (a - b)(a + b)$$

c) $x^2 - 2x - 35$

Answer:

$$(5)(-7) = -35, \quad (5) + (-7) = -2, \\ x^2 - 2x - 35 = (x + 5)(x - 7)$$

d) $3x^2 + 14x + 8$

Answer:

$$212 = 24, \quad 2 + 12 = 14, \\ 3x^2 + 14x + 8 = 3x^2 + 2x + 12x + 8 \\ = x(3x + 2) + 4(3x + 2) \\ = (3x + 2)(x + 4)$$

4. Determine the vertex of the following quadratic equations. (3 marks)

a) $f(x) = -3(x + 2)^2 + 9$

Answer: $(-2, 9)$

b) $g(x) = x^2 - 8$

Answer: $(0, -8)$

c) $h(x) = 7(x - 5)^2$

Answer: $(5, 0)$

5. Using the mapping formula, map the following point on the graph of $y = x^3$ onto the corresponding graph of $y = 5(x - 7)^3 - 2$. (8 marks: 2 marks

each)

a) (0, 0)

Answer:

Mapping Rule: $(x, y) \rightarrow (x + 7, 5y - 2)$

$(0, 0) \rightarrow (0 + 7, 5(0) - 2) = (7, -2)$

b) (1, 1)

Answer:

Mapping Rule: $(x, y) \rightarrow (x + 7, 5y - 2)$

$(1, 1) \rightarrow (1 + 7, 5(1) - 2) = (8, 3)$

c) (2, 8)

Answer:

Mapping Rule: $(x, y) \rightarrow (x + 7, 5y - 2)$

$(2, 8) \rightarrow (2 + 7, 5(8) - 2) = (9, 38)$

d) (-2, -8)

Answer:

Mapping Rule: $(x, y) \rightarrow (x + 7, 5y - 2)$

$(-2, -8) \rightarrow (-2 + 7, 5(-8) - 2) = (5, -42)$

Task 2: Thinking questions

6. Given the following table of values, determine the equation of the

polynomial function in standard form. (10 marks)

x	y
-4	-56
-3	-12
-2	0
-1	-8
0	-24
1	-36
2	-32
3	0
4	72

Answer:

Calculate finite differences to determine the degree of the polynomial:

x	y	1 difference	2 difference	3 difference
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-4	-56	44	-32	
-3	-12	12		12
-2	0	-8	-20	12
-1	-8	-16	-8	12
0	-24	-12	4	12
1	-36	4	16	12
2	-32	32	28	12
3	0	72	40	
4	72			

The third differences are constant: = 12. Therefore, deg = 3 and the table represents a cubic function. Determine the first coefficient:

$$a = \frac{12}{6} = 2.$$

From the table we can see that the cubic polynomial has two zeroes: $x = -2$ of order 2 (since the polynomial does not change sign at $x = -2$) and $x = 3$ of order 1. Therefore, the equation of the cubic polynomial is:

$$y = 2(x + 2)^2 (x - 3)$$

Expand and simplify the equation to standard form.

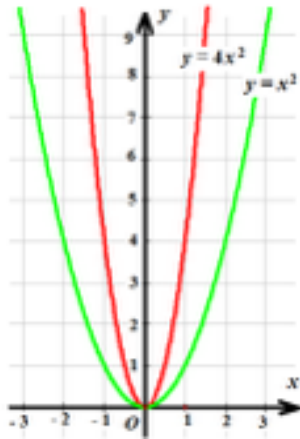
$$\begin{aligned}y &= 2(x^2 + 4x + 4)(x - 3) \\&= 2(x^3 + 4x^2 + 4x - 3x^2 - 12x - 12) \\&= 2(x^3 + x^2 - 8x - 12) \\&= 2x^3 + 2x^2 - 16x - 24\end{aligned}$$

The equation is $y = 2x^3 + 2x^2 - 16x - 24$.

7. a) Graph $y = x^2$ and $y = 4x^2$ on the same grid. Describe the transformation that has been applied to the graph of the parent function $y = x^2$ to get the graph of $y = 4x^2$. (2 marks)

Answer:

To get the graph of $y = 4x^2$, the graph of $y = x^2$ has been stretched vertically by a factor of 4. The mapping Rule is: $(x, y) \rightarrow (x, 4y)$.

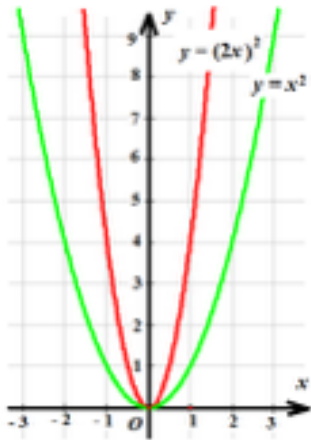


b) Graph $y = x^2$ and $y = (2x)^2$ on the same grid. Describe the transformation that has been applied to the graph of the parent function $y = x^2$ to get the graph of $y = (2x)^2$. (2 marks)

Answer:

To get the graph of $y = (2x)^2$, the graph of $y = x^2$ has been compressed

horizontally by a factor of 0.5. The mapping Rule is: $(x, y) \rightarrow (0.5x, y)$.



c) Show algebraically that the functions $f(x) = 4x^2$ and $g(x) = (2x)^2$ are equivalent. (3 marks)

Answer:

$$g(x) = (2x)^2 = (2^2)(x^2) = 4x^2 = f(x)$$

Hence, the two functions have the same graph.

8. For the polynomial function $f(x) = -5(2x + 1)(x - 2)^2(x - 6)$:

a) State the end behaviours of the graph. (3 marks)

Answer:

To state the end behaviours, determine the leading coefficient and the degree of the polynomial.

$$\text{Leading coefficient} = -5(2)(1^2)(1) = -10$$

$$\text{Degree} = 1 + 2 + 1 = 4$$

The leading coefficient is negative and the degree is even. Therefore, the end behaviours are:

$$\text{As } x \rightarrow \infty, f(x) \rightarrow -\infty.$$

$$\text{As } x \rightarrow -\infty, f(x) \rightarrow -\infty$$

b) Determine the roots of the polynomial. (3 marks)

Answer:

$$-5(2x + 1)(x - 2)^2(x - 6) = 0 \implies$$

$$2x + 1 = 0 \quad \text{or} \quad (x - 2)^2 = 0 \quad \text{or} \quad x - 6 = 0$$

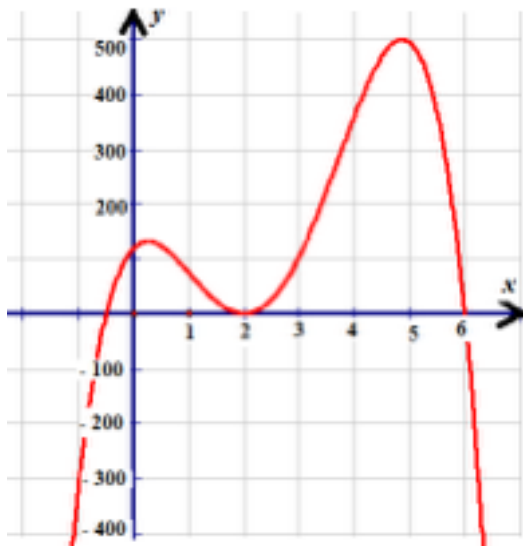
$$x = -0.5 \quad x = 2 \quad x = 6$$

(root of order 2)

c) Sketch a graph of the function. (2 marks)

Answer:

Graph of $f(x) = -5(2x + 1)(x - 2)^2(x - 6)$:



d) State the domain and range of the function. (2 marks)

Answer:

Domain = $\{ \}$, Range = $\{y \mid y \leq 500\}$.

Task 3: Communication questions

9. Define a term, an expression, and an equation, giving an example of each. What is the difference between an expression and an equation? (4 marks)

Answer:

A term is either a single number or variable, or numbers and variables multiplied together.

Examples: 5, x , $5x^2y^3$

An expression is a sequence of terms separated with plus and minus signs.

Examples: $x + 5$, $3x^2 + 2x - 1$, $7xy^3 + 4x^2y - xyz + 8$

An equation is two expressions separated by equal sign.

Examples: $3x + 2 = 5$, $2x^2 - 18 = 0$, $2x^3 = 128$

The difference between an expression and an equation is that an equation contains an equal sign, while an expression does not.

10. Determine whether the following polynomial functions are even, odd, or neither. Explain your reasoning. (6 marks: 2 marks each)

a) $f(x) = 3x^4 - 5x^2 + 8$

Answer:

The function is even since for all x holds $f(-x) = f(x)$:

$$f(-x) = 3(-x)^4 - 5(-x)^2 + 8 = 3x^4 - 5x^2 + 8 = f(x).$$

Note: An even polynomial function contains only a constant term or terms with variable x raised to even exponents.

b) $g(x) = 7x^3 - 11x$

Answer:

The function is odd since for all x holds $g(-x) = -g(x)$:

$$g(-x) = 7(-x)^3 - 11x = -7x^3 + 11x = -g(x).$$

Note: An odd polynomial function contains only terms with variable x raised to odd exponents.

c) $h(x) = 6x^5 - 9x^3 + 2$

Answer:

Neither:

$$h(-1) = 6(-1)^5 - 9(-1)^3 + 2 = 5,$$

$$h(1) = 6(1)^5 - 9(1)^3 + 2 = -1$$

$h(-1) \neq h(1) \implies$ the function is not even

$h(-1) \neq -h(1) \implies$ the function is not odd

11. Explain how to find the domain and range of a polynomial function.
(6 marks)

Answer:

The domain of all polynomial functions is the set of all real numbers: $\{x \in \mathbb{R}\}$. The range of all polynomial functions with odd degree is the set of all real numbers: $\{y \in \mathbb{R}\}$. The range of all polynomial functions with even degree and positive leading coefficient is $\{y \in \mathbb{R} \mid y \geq \text{abs min}\}$, where *abs min* is the absolute minimum of the function on the set of all real numbers. The range of all polynomial functions with even degree and negative leading coefficient is $\{y \in \mathbb{R} \mid y \leq \text{abs max}\}$, where *abs max* is the absolute maximum of the function on the set of all real numbers.

12. State the value of a , k , c , and d for the following equation and describe the transformations that apply: $f(x) = -(x + 2)^3 - 7$. (8 marks)

Answer:

$$a = -, k = 1, c = -7, d = -2.$$

The parent function is $y = x^3$.

Transformations:

- Vertical compression by a factor of :

$$y = x^3 \quad y = \frac{1}{2}x^3$$

- Reflection in the x -axis:

$$y = x^3 \quad y = -x^3$$

- Horizontal translation of 2 units to the left:

$$y = x^3 \quad y = (x + 2)^3$$

- Vertical translation of 7 units downwards:

$$y = x^3 \quad y = x^3 - 7$$

Task 4: Application questions

13. A potter sells bowls for \$18 each. The equation that models revenue is $R(x) = 18x$, where $R(x)$ represents the total revenue (in dollars), and x represents the number of bowls sold. The potter's set-up fees for making the bowls are \$132, and the bowls cost \$6 each to make. The equation that models the cost of the bowls is $C(x) = 6x + 132$, where $C(x)$ represents the total cost and x represents the number of bowls sold.

a) How many bowls must the potter sell in order to break even? What is the cost of making this many bowls? Show your calculations

algebraically. (Hint: The break-even point is where the revenue equals the cost.) (3 marks)

Answer:

$$R(x) = C(x) \implies 18x = 6x + 132 \implies 12x = 132 \implies x = 11$$

$$C(11) = 6(11) + 132 = 198.$$

The potter must sell 11 bowls to break even. The cost of making the bowls is \$198.

b) How many bowls must the potter sell in order to make a profit? (1 mark)

Answer:

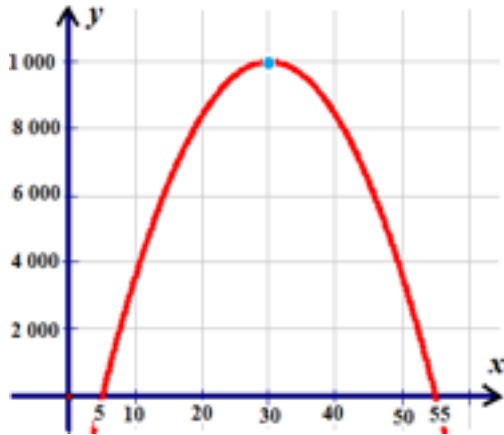
The potter must sell at least 12 bowls in order to make a profit, because $R(x) > C(x)$ for $x > 11$.

14. To maximize yearly revenue, a company wishes to determine the best selling price for their new line of shirts. They realize that, if the price is too low or too high, they will not sell as many shirts. The quadratic function $y = -16(x - 30)^2 + 10\,000$ models this scenario, where x represents the selling price per shirt and y represents the number of shirts sold.

a) Sketch a graph of the function, indicating the vertex and x -intercepts. Show how you determined the x -intercepts algebraically. (4 marks)

Answer:

Number of shirts sold



Selling price per shirt

The vertex of the parabola is (30, 10 000).

Determine the x -intercepts:

$$y = 0 \implies -16(x - 30)^2 + 10\,000 \quad \text{Divide by } -16$$

$$(x - 30)^2 - 625 = 0 \quad \text{Factor as difference of squares}$$

$$[(x - 30) - 25][(x - 30) + 25] = 0$$

$$(x - 55)(x - 5) = 0$$

The x -intercepts are (5, 0) and (55, 0).

Shirt sales

Selling price per shirt

b) Determine the selling price that will maximize revenue. (1 mark)

Answer:

The selling price that will maximize revenue is \$30. This is represented by the x -value of the vertex.

c) Determine the number of shirts sold yearly at this selling price. (1 mark)

Answer:

At \$30 per shirt, there will be 10 000 shirts sold annually. This is

represented by the y -value of the vertex.

d) What is the maximum yearly revenue? (1 mark)

Answer:

The yearly revenue is the price per shirt multiplied by the number of shirts sold.

$$\text{Revenue} = 30(10\,000) = \$300\,000$$

The maximum yearly revenue is \$300 000.

e) What do the x -intercepts represent? (1 mark)

Answer:

At the x -intercepts, the y -value (which represents the number of shirts sold) is 0. This means that there will be no shirts sold when the price per shirt is \$5 or \$55.

15. The dimensions of a box are 8 cm \times 10 cm \times 12 cm. The width, length, and height of the box must be increased by the same amount, x cm, in order for the box to have a volume of 2688 cm³.

a) Express the volume of the new box as a function $V(x)$ of variable x . (3 marks)

Answer:

The new dimensions of the box are $(x + 8)$, $(x + 10)$, and $(x + 12)$ cm. Hence, the volume of the new box is

$$\begin{aligned} V(x) &= (x + 8)(x + 10)(x + 12) \\ &= (x^2 + 18x + 80)(x + 12) \\ &= x^3 + 18x^2 + 80x + 12x^2 + 216x + 960 \\ &= x^3 + 30x^2 + 296x + 960 \end{aligned}$$

The volume of the box (in cm³) is $V = x^3 + 30x^2 + 296x + 960$, where x cm represents the amount that must be added to each dimension.

b) Determine the value of x . (5 marks)

Answer:

Substitute $V = 2688$ and solve the equation:

$$2688 = x^3 + 30x^2 + 296x + 960$$

$$x^3 + 30x^2 + 296x - 1728 = 0$$

Factor the equation using the factor theorem:

$$x = 4: 4^3 + 30(4^2) + 296(4) - 1728 = 0$$

Since $x = 4$ is a root, $(x - 4)$ is a factor. Use long division to find the other factors.

$$\begin{array}{r} x^2 + 34x + 432 \\ x - 4 \overline{) x^3 + 30x^2 + 296x - 1728} \\ \underline{x^3 - 4x^2} \\ 34x^2 + 296x \\ \underline{34x^2 - 136x} \\ 432x - 1728 \\ \underline{432x - 1728} \\ 0 \end{array}$$

$$x^3 + 30x^2 + 296x - 1728 = (x - 4)(x^2 + 34x + 432)$$

The quadratic trinomial $x^2 + 34x + 432$ does not have roots, since its discriminant $D = b^2 - 4ac$ is negative:

$$D = 34^2 - 4(1)(432) = 1156 - 1728 < 0.$$

The only real root for the cubic function is $x = 4$. Therefore, 4 cm must be added to each dimension of the box.

c) What dimensions will create a box with a volume of 2688 cm^3 ? (2 marks)

Answer:

The dimensions of the box are

$$x + 8 = 4 + 8 = 12,$$

$$x + 10 = 4 + 10 = 14,$$

$$x + 12 = 4 + 12 = 16.$$

The dimensions of the new box are 12 cm \times 14 cm \times 16 cm.