1 a) Recall law of total probability which says

$$P(A) = P(A/B)\cdot P(B) + P(A/B^c)\cdot P(B^c)$$

Hence

$$P(Y=r) = P(Y=r/\text{log is oak})P(\text{log is oak}) + P(Y=r/\text{log is maple})P(\text{log is maple})$$

$$= \tfrac{1}{3}\left(\tfrac{2}{3}\right)^{r-1} \times \tfrac{3}{4} + \tfrac{2}{3}\left(\tfrac{1}{3}\right)^{r-1} \times \tfrac{1}{4}$$

$$= \tfrac{1}{4}\left(\tfrac{2}{3}\right)^{r-1} + \tfrac{1}{6}\left(\tfrac{1}{3}\right)^{r-1}$$

$$= \tfrac{1}{6}\left(\tfrac{1}{3}\right)^{r-1} + \tfrac{1}{4}\left(\tfrac{2}{3}\right)^{r-1} \quad , r = 1, 2, 3 \ldots$$

b) $$E(Y) = \sum_{r=1}^{\infty} r\,P(Y=r)$$

$$= \sum_{r=1}^{\infty} r\,\tfrac{1}{4}\left(\tfrac{2}{3}\right)^{r-1} + \sum_{r=1}^{\infty} r\,\tfrac{1}{6}\left(\tfrac{1}{3}\right)^{r-1}$$

$$= \tfrac{1}{4}\sum_{r=1}^{\infty} r\left(\tfrac{2}{3}\right)^{r-1} + \tfrac{1}{6}\sum_{r=1}^{\infty} r\left(\tfrac{1}{3}\right)^{r-1}$$

$$= \tfrac{1}{4} \times 3 \times \underbrace{\sum_{r=1}^{\infty} \tfrac{1}{3}(r)\left(\tfrac{2}{3}\right)^{r-1}}_{} + \tfrac{1}{6} \times \tfrac{3}{2} \times \underbrace{\sum_{r=1}^{\infty} \tfrac{2}{3} r \left(\tfrac{1}{3}\right)^{r-1}}_{}$$

Note this is $E(X)$ for $X \sim \text{Geometric}(\tfrac{1}{3})$ $\Rightarrow E(X) = 3$

$Y \sim \text{Geometric}(\tfrac{2}{3})$ $E(Y) = \tfrac{3}{2}$

$$= \tfrac{1}{4} \times 3 \times 3 + \tfrac{1}{6} \times \tfrac{3}{2} \times \tfrac{3}{2}$$

$$= \tfrac{9}{4} + \tfrac{3}{8}$$

$$= \frac{18+3}{8} = \frac{21}{8}$$

(c) $$P(Y \le 4 \mid Y > 2) = \frac{P(Y \le 4 \cap Y > 2)}{P(Y > 2)}$$

$$= \frac{P(Y=3) + P(Y=4)}{1 - [P(Y=1) + P(Y=2)]}$$

note that $P(Y=0) = 0$ since $Y$ can only take on values $1, 2, 3 \ldots$

$$= \frac{\frac{1}{6}\left(\frac{1}{3}\right)^2 + \frac{1}{4}\left(\frac{2}{3}\right)^2 + \frac{1}{6}\left(\frac{1}{3}\right)^3 + \frac{1}{4}\left(\frac{2}{3}\right)^3}{1 - \left(\frac{1}{6} + \frac{1}{4}\right) - \left[\frac{1}{6}\left(\frac{1}{3}\right) + \frac{1}{4}\left(\frac{2}{3}\right)\right]}$$

$$= \frac{\frac{1}{54} + \frac{1}{9} + \frac{1}{162} + \frac{2}{27}}{1 - \frac{10}{24} - \frac{4}{18}}$$

$$= \frac{\frac{6+36+2+24}{324}}{\frac{324-135-72}{324}}$$

$$= \frac{68}{117}$$

2. Let $X$ be the random variable that represents the number of lights that needs to be replaced.

$X \sim \text{Poisson}(2.2)$ [in any 1 day period]

(a) $P(X=0)$

$= \dfrac{e^{-2.2}(2.2)^0}{0!}$

$= e^{-2.2}$

$= 0.1108$

(b) $P(X \geq 4)$

$= 1 - P(X \leqslant 3)$

$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$

$= 1 - \left[ e^{-2.2} + \dfrac{e^{-2.2}(2.2)^1}{1!} + \dfrac{e^{-2.2}(2.2)^2}{2!} + \dfrac{e^{-2.2}(2.2)^3}{3!} \right]$

$\approx 0.1806$

(c) Let $N$ be the number of consecutive days

$X \sim \text{Poisson}(2.2N)$ [in any $N$ consecutive days]

We wish to find the least value of $N$ such that

$P(X \geq 1) > 0.9999$

$1 - P(X=0) > 0.9999$

$1 - e^{-2.2N} > 0.9999$

$e^{-2.2N} < 0.0001$

Take ln both sides

$-2.2N < \ln(0.0001)$

$-2.2N < -9.2103$

$N > 4.18 \Rightarrow$ the least value of $N$ is 5 (since $N$ must be an integer)

(d) From part b, we know that

P( at least 4 lights will need to be replaced in a particular day)

$= 0.1806$

Let $Y$ be the random variable that represents the number of days out of 7 that require "4 lights to be replaced in a particular day".

Then $Y \sim \text{Binomial}(7, 0.1806)$

$P(Y \geq 2)$

$= 1 - [P(Y=0) + P(Y=1)$

$= 1 - [{}^7C_0 (0.1806)^0 (0.8194)^7 + {}^7C_1 (0.1806)^1 (0.8194)^6]$

$\approx 0.369$

3. Let $X$ be the length of time for which an ordinary light bulb will last

$$X \sim \text{Normal}(600, 100^2).$$

Let $Y$ be the length of time for which a new long-life bulb will last

$$Y \sim \text{Normal}(2000, 200^2)$$

(a) $P(X > 450) = P(Z > \frac{450-600}{100})$

$= P(Z > -1.5)$

$= 1 - P(Z < -1.5)$

$= 1 - 0.0668$

$= 0.9332$

(b) Let the lifetimes of two ordinary bulbs be $X_1$ and $X_2$ (students who use $2X$ have the wrong concept).

$X_1 + X_2 \sim N(1200, 20000)$ because sum of independent normal is itself normal and $Var(X_1 + X_2) = Var(X_1) + Var(X_2)$ because of independence

$P(X_1 + X_2 < 1100) = P(Z < \frac{1100-1200}{\sqrt{20000}})$

$= P(Z < -0.707)$

$= 0.2389$

(c) Let $\bar{X}$ be the sample mean of 100 independent observations of X.

$$\bar{X} \sim N\left(600, \frac{100^2}{100}\right)$$

$$\Rightarrow \bar{X} \sim N(600, 100)$$

$$P(\bar{X} > 595) = P\left(Z > \frac{595-600}{\sqrt{100}}\right)$$

$$= P(Z > -0.5)$$

$$= 1 - P(Z < -0.5)$$

$$= 1 - 0.3085$$

$$= 0.6915$$

(d) We wish to find $P(Y > 3X)$

$$P(Y > 3X) = P(Y - 3X > 0)$$

$$E(Y-3X) = E(Y) - 3E(X)$$

$$= 2000 - 3(600)$$

$$= 200$$

$$Var(Y-3X) = var(Y) + 3^2 var(X)$$

$$= 200^2 + 9(100^2)$$

$$= 130000$$

$Y - 3X \sim N(200, 130000)$ since sum/differences of independent normal is itself normal

$$P(Y-3X > 0) = P\left(Z > \frac{0-200}{\sqrt{130000}}\right)$$

$$= P(Z > -0.555)$$

$$= 1 - P(Z < -0.555)$$

$$= 1 - 0.28945$$

$$\approx 0.7106$$

4. Let $X$ be the random variable which represents the number of defective articles among 9 articles.

Then $X \sim \text{Binomial}(9, p)$.

Conditions for accepting and rejecting the batch are

(a) $X \geq 2$, reject batch

(b) $X < 2$, accept batch

$P(\text{batch is accepted})$

$= P(X<2)$

$= P(X=0) + P(X=1)$

$= {}^9C_0 (p)^0 (1-p)^{9-0} + {}^9C_1 (p)^1 (1-p)^{9-1}$

$= (1-p)^9 + 9p(1-p)^8$

$= (1-p)^8 \left[(1-p) + 9p\right]$

$= (1-p)^8 (1+8p)$

Let $Y$ be the random variable which represents the number of defective articles in the second sample of 9 articles.

Then $Y \sim \text{Binomial}(9, p)$

For this new scheme, the conditions for accepting and rejecting the batches are :-

(a) $X = 1$ and $Y = 0$, accept batch

(b) $X < 1$, also accept batch

(c) otherwise, reject batch

P( batch is accepted)

$= P(X=1, Y=0) + P(X<1)$

$= P(X=1)P(Y=0) + P(X<1)$ (since X and Y are independent)

$= {}^9C_1 (p)^1 (1-p)^8 \cdot {}^9C_0 (p)^0 (1-p)^9 + P(X=0)$

$= 9p(1-p)^8 \cdot (1-p)^9 + (1-p)^9$

$= (1-p)^9 [\, 9p(1-p)^8 + 1 \,]$

Under modified scheme, 9 articles are examined in the first batch and 9 articles will be examined in the second batch if $P(X=1)$.

Hence, averge number sampled per manufactured batch

$= 9 + [P(X=1) \times 9]$

If you cannot see this, you need to work it out this way.

Let W be the number of articles examined in the second batch

| W | 0 | 9 |
|---|---|---|
| prob. | $1-P(X=1)$ | $P(X=1)$ |

$\therefore E(W) = 0 \times [1-P(X=1)] + 9 \times P(X=1)$
$= 9P(X=1)$

$= 9 + [9p(1-p)^8 \times 9]$

$= 9 + 81p(1-p)^8$

$= 9 + 81(0.1)(0.9)^8.$

$\approx 12.49$