Chapter 11

Angular Momentum

Angular Momentum

Angular momentum plays a key role in rotational dynamics.

There is a principle of conservation of angular momentum.

- In analogy to the principle of conservation of linear momentum
- The angular momentum of an isolated system is constant.
  - For angular momentum, an isolated system is one in which no external torques act on the system.

The law of conservation of angular momentum is a fundamental law of physics.

- Also valid for relativistic and quantum systems

The Vector Product

There are instances where the product of two vectors is another vector.

- Earlier we saw where the product of two vectors was a scalar.
  - This was called the dot product.

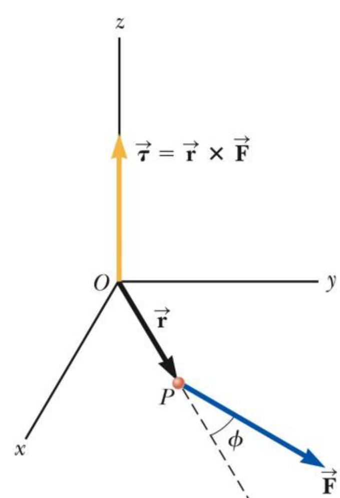
The vector product of two vectors is called the cross product.

The Vector Product and Torque

The torque vector lies in a direction perpendicular to the plane formed by the position vector and the force vector.

$\vec{\tau} = \vec{\mathbf{F}} \times \vec{\mathbf{r}}$

The torque is the vector (or cross) product of the position vector and the force vector.



The Vector Product Defined

Given two vectors, $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$

The vector (cross) product of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined as a *third vector*, $\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$.

- **C** is read as "**A** cross **B**".

The magnitude of vector C is *AB* sin θ.

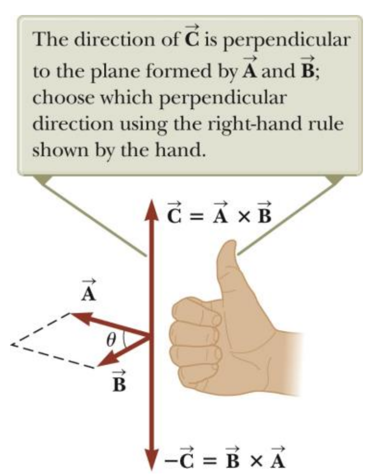
- θ is the angle between $\vec{\mathbf{A}}$ and. $\vec{\mathbf{B}}$

More About the Vector Product

The quantity *AB* sin θ is equal to the area of the parallelogram formed by $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

The direction of $\vec{\mathbf{C}}$ is perpendicular to the plane formed by $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

The best way to determine this direction is to use the right-hand rule.



Properties of the Vector Product

The vector product is *not* commutative. The order in which the vectors are multiplied is important.

- To account for order, remember $\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}$

If $\vec{\mathbf{A}}$ is parallel to $\vec{\mathbf{B}}$ ($\theta = 0^{o}$ or 180^{o}), then $\vec{\mathbf{A}} \times \vec{\mathbf{B}} = 0$

- Therefore $\vec{\mathbf{A}} \times \vec{\mathbf{A}} = 0$

If $\vec{\mathbf{A}}$ is perpendicular to $\vec{\mathbf{B}}$, then $\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = AB$

The vector product obeys the distributive law.

$$\vec{\mathbf{A}} \times (\vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \vec{\mathbf{C}}$$

Final Properties of the Vector Product

The derivative of the cross product with respect to some variable such as *t* is

$$\frac{d}{dt}\left(\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right) = \frac{d\vec{\mathbf{A}}}{dt} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \frac{d\vec{\mathbf{B}}}{dt}$$

where it is important to preserve the multiplicative order of the vectors.

Vector Products of Unit Vectors

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0$$
$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = -\hat{\mathbf{j}} \times \hat{\mathbf{i}} = \hat{\mathbf{k}}$$
$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = -\hat{\mathbf{k}} \times \hat{\mathbf{j}} = \hat{\mathbf{i}}$$
$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = -\hat{\mathbf{i}} \times \hat{\mathbf{k}} = \hat{\mathbf{j}}$$

Signs in Cross Products

Signs are interchangeable in cross products

- $\vec{\mathbf{A}} \times \left(-\vec{\mathbf{B}}\right) = -\vec{\mathbf{A}} \times \vec{\mathbf{B}}$
- and $\hat{\mathbf{i}} \times \left(-\hat{\mathbf{j}}\right) = -\hat{\mathbf{i}} \times \hat{\mathbf{j}}$

Using Determinants

The cross product can be expressed as

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \hat{\mathbf{i}} + \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \hat{\mathbf{k}}$$

Expanding the determinants gives

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \left(A_y B_z - A_z B_y\right) \hat{\mathbf{i}} + \left(A_x B_z - A_z B_x\right) \hat{\mathbf{j}} + \left(A_x B_y - A_y B_x\right) \hat{\mathbf{k}}$$

Vector Product Example

Given $\vec{\mathbf{A}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}};\ \vec{\mathbf{B}} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$

Find $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$

Result

$$\begin{aligned} \vec{\mathbf{A}} \times \vec{\mathbf{B}} &= (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) \times (-\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) \\ &= 2\hat{\mathbf{i}} \times (-\hat{\mathbf{i}}) + 2\hat{\mathbf{i}} \times 2\hat{\mathbf{j}} + 3\hat{\mathbf{j}} \times (-\hat{\mathbf{i}}) + 3\hat{\mathbf{j}} \times 2\hat{\mathbf{j}} \\ &= 0 + 4\hat{\mathbf{k}} + 3\hat{\mathbf{k}} + 0 = 7\hat{\mathbf{k}} \end{aligned}$$

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Torque Vector Example

Given the force and location

$$\vec{\mathbf{F}} = (2.00\,\hat{\mathbf{i}} + 3.00\,\hat{\mathbf{j}})\ N$$

$$\vec{\mathbf{r}} = (4.00\,\hat{\mathbf{i}} + 5.00\,\hat{\mathbf{j}})\ m$$

Find the torque produced

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = [(4.00\hat{\mathbf{i}} + 5.00\hat{\mathbf{j}})\text{N}] \times [(2.00\hat{\mathbf{i}} + 3.00\hat{\mathbf{j}})\text{m}]$$

$$= [(4.00)(2.00)\hat{\mathbf{i}} \times \hat{\mathbf{i}} + (4.00)(3.00)\hat{\mathbf{i}} \times \hat{\mathbf{j}}$$

$$+(5.00)(2.00)\hat{\mathbf{j}} \times \hat{\mathbf{i}} + (5.00)(3.00)\hat{\mathbf{i}} \times \hat{\mathbf{j}}$$

$$= 2.0\hat{\mathbf{k}}\ \text{N}\cdot\text{m}$$

Angular Momentum

Consider a particle of mass *m* located at the vector position $\vec{\mathbf{r}}$ and moving with linear momentum $\vec{\mathbf{p}}$.

Find the net torque.

$$\vec{\mathbf{r}} \times \sum \vec{\mathbf{F}} = \sum \vec{\tau} = \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt}$$

Add the term $\frac{d\vec{\mathbf{r}}}{dt} \times \vec{\mathbf{p}}\,(\text{since it} = 0)$

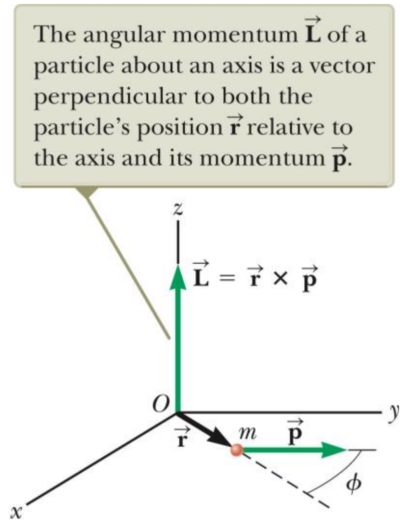
$$\sum \vec{\tau} = \frac{d(\vec{\mathbf{r}} \times \vec{\mathbf{p}})}{dt}$$

This looks very similar to the equation for the net force in terms of the linear momentum since the torque plays the same role in rotational motion that force plays in translational motion.

Angular Momentum, cont

The instantaneous angular momentum of a particle relative to the origin *O* is defined as the cross product of the particle's instantaneous position vector and its instantaneous linear momentum.

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$$



Torque and Angular Momentum

The torque is related to the angular momentum.

- Similar to the way force is related to linear momentum.

$$\sum \vec{\tau} = \frac{d\vec{\mathbf{L}}}{dt}$$

The torque acting on a particle is equal to the time rate of change of the particle's angular momentum.

This is the rotational analog of Newton's Second Law .

- $\sum \vec{\tau}$ and $\vec{\mathbf{L}}$ must be measured about the same origin.
- This is valid for any origin fixed in an inertial frame.

More About Angular Momentum

The SI units of angular momentum are (kg·m^2)/ s.

Both the magnitude and direction of the angular momentum depend on the choice of origin.

The magnitude is $L = mvr \sin \phi$

- ϕ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$.

The direction of $\vec{\mathbf{L}}$ is perpendicular to the plane formed by $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$.

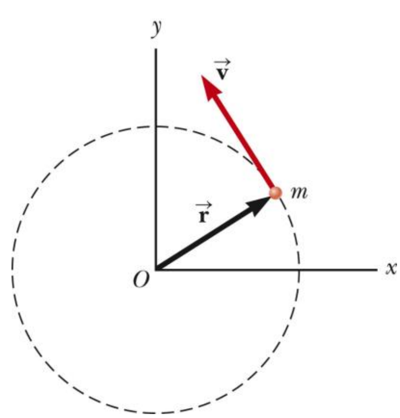
Angular Momentum of a Particle, Example

The vector $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$ is pointed out of the diagram.

The magnitude is $L = mvr \sin 90^\circ = mvr$

- $\sin 90^\circ$ is used since v is perpendicular to r.

A particle in uniform circular motion has a constant angular momentum about an axis through the center of its path.



Angular Momentum of a System of Particles

The total angular momentum of a system of particles is defined as the vector sum of the angular momenta of the individual particles.

- $\vec{\mathbf{L}}_{tot} = \vec{\mathbf{L}}_1 + \vec{\mathbf{L}}_2 + \ldots + \vec{\mathbf{L}}_n = \sum_i \vec{\mathbf{L}}_i$

Differentiating with respect to time

$$\frac{d\vec{\mathbf{L}}_{\text{tot}}}{dt} = \sum_i \frac{d\vec{\mathbf{L}}_i}{dt} = \sum_i \vec{\tau}_i$$

Angular Momentum of a System of Particles, cont

Any torques associated with the internal forces acting in a system of particles are zero.

Therefore, $\sum \vec{\tau}_{\text{ext}} = \frac{d\vec{\mathbf{L}}_{\text{tot}}}{dt}$

- The net external torque acting on a system about some axis passing through an origin in an inertial frame equals the time rate of change of the total angular momentum of the system about that origin.

This is the mathematical representation of the angular momentum version of the non-isolated system model.

Rearranging the equation gives $\int \left(\sum \vec{\tau}_{ext} \right) dt = \Delta \vec{\mathbf{L}}_{tot}$.

This is the angular impulse-angular momentum theorem.

Angular Momentum of a System of Particles, final

The resultant torque acting on a system about an axis through the center of mass equals the time rate of change of angular momentum of the system regardless of the motion of the center of mass.

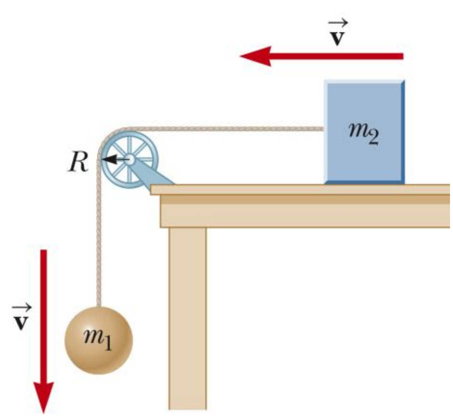
- This applies even if the center of mass is accelerating, provided $\vec{\tau}$ and $\vec{\mathbf{L}}$ are evaluated relative to the center of mass.

System of Objects, Example

The masses are connected by a light cord that passes over a pulley; find the linear acceleration.

Conceptualize

- The sphere falls, the pulley rotates and the block slides.
- Use concepts of angular momentum and torque.



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System of Objects, Example cont.

Categorize

- The block, pulley and sphere are a non-isolated system.
  - Subject to an external torque due to the gravitational force on the sphere.
- Use an axis that corresponds to the axle of the pulley.
- The angular momentum of the system includes two objects moving translationally and one undergoing pure rotation.

Analyze

- At any instant of time, the sphere and the block have a common velocity *v*.
- Write expressions for the total angular momentum and the net external torque.
- Solve the expression for the linear acceleration.

System of Objects, Example final

Finalize

- The system as a whole was analyzed so that internal forces could be ignored.
- Only *external* forces are needed.
  - Only external forces contribute to the change in the system's angular momentum.

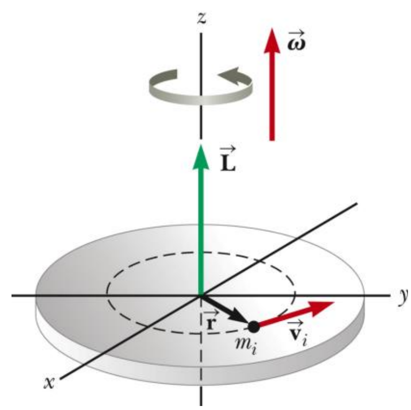
Angular Momentum of a Rotating Rigid Object

The rigid object is a non-deformable system.

Each particle of the object rotates in the *xy* plane about the *z* axis with an angular speed of ω.

The angular momentum of an individual particle is $L_i = m_i r_i^2 \omega$.

$\vec{\mathbf{L}}$ and $\vec{\omega}$ are directed along the *z* axis.



Angular Momentum of a Rotating Rigid Object, cont

To find the angular momentum of the entire object, add the angular momenta of all the individual particles.

$$L_z = \sum_i L_i = \sum_i \left(m_i r_i^2\right)\omega = I\omega$$

Differentiating, this gives the rotational form of Newton's Second Law.

$$\sum \vec{\tau}_{\text{ext}} = \frac{dL_z}{dt} = I\frac{d\omega}{dt} = I\alpha$$

This is the mathematical representation of the rigid object under a net torque analysis model.

Angular Momentum of a Rotating Rigid Object, final

The rotational form of Newton's Second Law is also valid for a rigid object rotating about a moving axis provided the moving axis:

- (1) passes through the center of mass
- (2) is a symmetry axis

If a symmetrical object rotates about a fixed axis passing through its center of mass, the vector form holds: $\vec{\mathbf{L}} = I\vec{\omega}$

- $\vec{\mathbf{L}}$ is the total angular momentum measured with respect to the axis of rotation.

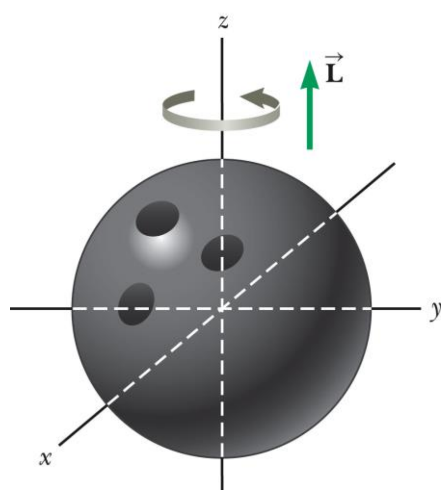
The rotational form of Newton's Second Law is also valid for any object, regardless of its symmetry, if $\vec{\mathbf{L}}$ stands for the component of angular momentum along the axis of rotation.

Angular Momentum of a Bowling Ball

The momentum of inertia of the ball is $2/5MR^2$.

The angular momentum of the ball is $L_z = I\omega$.

The direction of the angular momentum is in the positive z direction.



Conservation of Angular Momentum

The total angular momentum of a system is constant in both magnitude and direction if the net external torque acting on the system is zero.

- Net torque = 0 means that the system is isolated.
- This is the basis of the angular momentum version of the isolated system model.

$\vec{\mathbf{L}}_{tot}$ = constant or $\vec{\mathbf{L}}_i = \vec{\mathbf{L}}_f$

For a system of particles,

$$\vec{\mathbf{L}}_{tot} = \sum \vec{\mathbf{L}}_n = \text{constant}$$

Conservation of Angular Momentum, cont

If the system is deformable such that the mass of the isolated system undergoes redistribution, the moment of inertia changes.

- The conservation of angular momentum requires a compensating change in the angular velocity.
- $I_i \omega_i = I_f \omega_f$ = constant
  - This holds for rotation about a fixed axis and for rotation about an axis through the center of mass of a moving system.
  - The net torque must be zero in any case.

Conservation Law Summary

For an isolated system -

(1) Conservation of Energy:

- $E_i = E_f$
- If there is no energy transfers across the system boundary

(2) Conservation of Linear Momentum:

- $\vec{\mathbf{p}}_i = \vec{\mathbf{p}}_f$
- If the next external force on the system is zero

(3) Conservation of Angular Momentum:

- $\vec{\mathbf{L}}_i = \vec{\mathbf{L}}_f$
- If the next external force on the system is zero

Conservation Laws – Notes

A system may be isolated in terms of one of these quantities but not in terms of another.

- For example, a system is often non-isolated in terms of momentum and also not non-isolated in terms of energy because a net force or torque acts on it.
- Systems can be non-isolated in terms of energy but isolated in terms of momentum.
- Collisions are often isolated in terms of momentum but not in terms of energy.

Conservation of Angular Momentum: The Merry-Go-Round

The moment of inertia of the system is the moment of inertia of the platform plus the moment of inertia of the person.

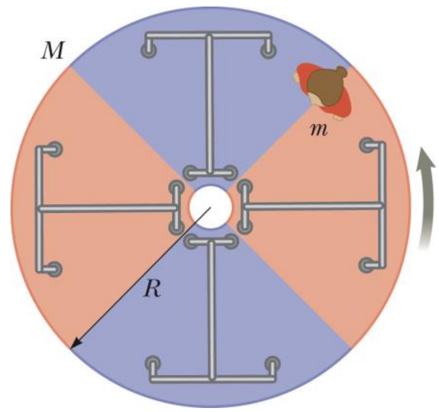
- Assume the person can be treated as a particle.

As the person moves toward the center of the rotating platform, the angular speed will increase.

- To keep the angular momentum constant

The system is isolated in terms of angular momentum.

- The system is isolated in terms of energy, but potential energy changes to kinetic energy.



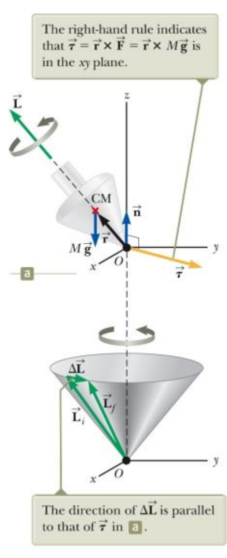
Motion of a Top

The only external forces acting on the top are the normal force and the gravitational force .

The direction of the angular momentum is along the axis of symmetry.

The right-hand rule indicates that the torque is in the *xy* plane.

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \vec{\mathbf{r}} \times M\vec{\mathbf{g}}$$



Motion of a Top, cont

The net torque and the angular momentum are related:

$$\sum \vec{\tau}_{ext} = \frac{d\vec{\mathbf{L}}}{dt}$$

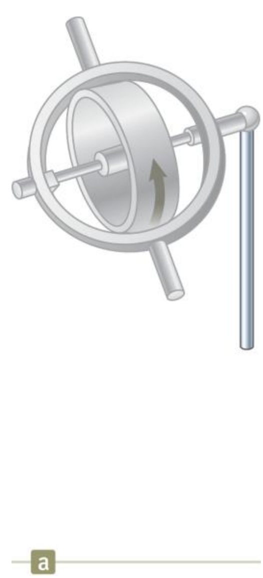
- A non-zero torque produces a change in the angular momentum.
- The result of the change in angular momentum is a precession about the z axis.
- The direction of the angular momentum is changing.
- The **precessional motion** is the motion of the symmetry axis about the vertical.
- The precession is usually slow relative to the spinning motion of the top.

Gyroscope

A gyroscope can be used to illustrate precessional motion.

The gravitational force produces a torque about the pivot, and this torque is perpendicular to the axle.

The normal force produces no torque.



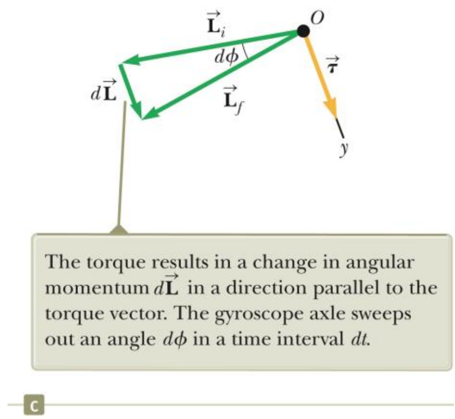
Gyroscope, cont

The torque results in a change in angular momentum in a direction perpendicular to the axle.

- The axle sweeps out an angle $d\phi$ in a time interval *dt*.

The direction, not the magnitude, of the angular momentum is changing .

The gyroscope experiences precessional motion.



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Gyroscope, final

To simplify, assume the angular momentum due to the motion of the center of mass about the pivot is zero.

- Therefore, the total angular momentum is due to its spin.
- This is a good approximation when $\vec{\omega}$ is large.

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Precessional Frequency

Analyzing the vector triangle in fig. 11.14 c, the rate at which the axle rotates about the vertical axis can be found.

$$\omega_p = \frac{d\phi}{dt} = \frac{Mgr_{CM}}{I\omega}$$

- ω_p is the **precessional frequency**
  - This is valid only when $\omega_p << \omega$

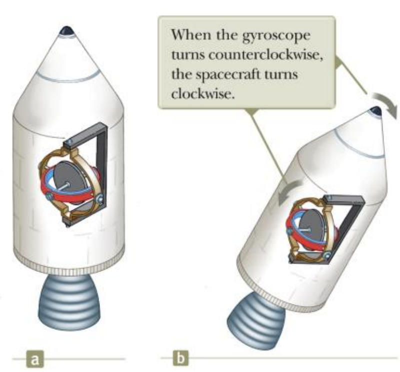
Gyroscope in a Spacecraft

The angular momentum of the spacecraft about its center of mass is zero.

A gyroscope is set into rotation, giving it a nonzero angular momentum.

The spacecraft rotates in the direction opposite to that of the gyroscope.

So the total momentum of the system remains zero.

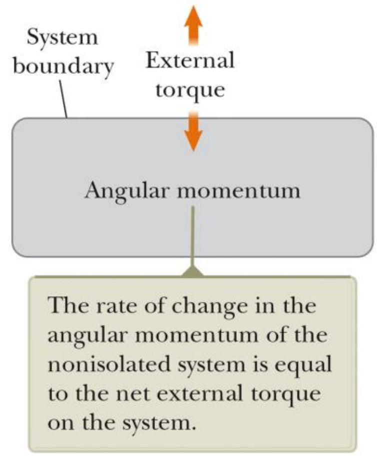


New Analysis Model – Nonisolated System

Nonisolated System (Angular Momentum)

- If a system interacts with its environment in the sense that there is an external torque on the system, the net external torque acting on the system is equal to the time rate of change of its angular momentum:

$$\sum \vec{\tau}_{ext} = \frac{d\vec{\mathbf{L}}_{tot}}{dt}$$



New Analysis Model – Isolated System

Isolated System (Angular Momentum)

- If a system experiences no external torque from the environment, the total angular momentum of the system is conserved.
  - $\vec{\mathbf{L}}_i = \vec{\mathbf{L}}_f$

Applying this law of conservation of angular momentum to a system whose moment of inertia changes gives

- $I_i\omega_i = I_f\omega_f = constant$