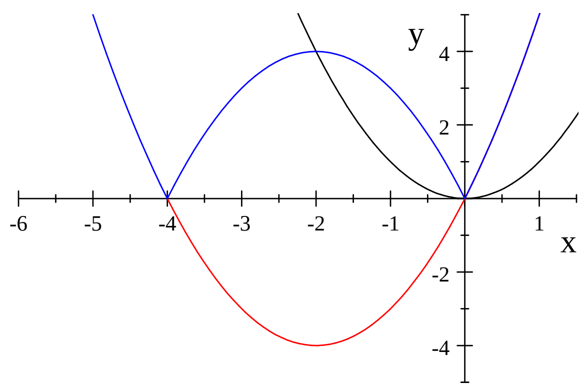
MATH 203 Final Exam April 2008
Solutions

1. (a) Sketch the graph of the function $f(x) = |(x+2)^2 - 4|$, starting from the graph

of the standard parabola and using appropriate transformations.

Solution



(b) Suppose $f(x) = \sqrt[3]{1+e^x}$ and $g(x) = \ln(x^3 - 1)$. Find $f \circ g$ and $g \circ f$. Determine the domain and range of $f \circ g$ and $g \circ f$.

Solution $f \circ g\,(x) = \sqrt[3]{1 + e^{\ln(x^3-1)}} = \sqrt[3]{1 + x^3 - 1} = x$ and $g \circ f\,(x) = \ln\left(\left(\sqrt[3]{1+e^x}\right)^3 - 1\right) = \ln(e^x) = x$. So f and g are inverse to each other which means Domain(f) = Range(g) and vice-versa. Now f is defined for all x, so Domain(f) = Range(g) = $\mathbb{R}$. On the other hand g is only defined if $x^3 - 1 > 0$, i.e. $x > 1$ so Domain(g) = Range(f) = $(1, \infty)$.

(c) Evaluate $\cos(\sin^{-1}(t))$.

Solution Let $u = \sin^{-1}(t)$. Then $\sin u = t$ and so since $\sin^2 u + \cos^2 u = 1$, $\cos^2 u = 1 - t^2$. So $\cos u = \cos(\sin^{-1}(t)) = \sqrt{1-t^2}$ with the + sign because $-\pi/2 \le u \le \pi/2$

2. Evaluate the limits:

(a) $\lim\limits_{x\to 4} \dfrac{\sqrt{2x+1} - 3}{x^3 - 64}$; (b) $\lim\limits_{x\to\infty} \dfrac{(x^4+1)(3-2x)^3}{(x+1)^5 (2-x^2)}$

Do not use l'Hopital's rule.

Solution (a)

$$\begin{aligned}
\lim_{x\to 4}\frac{\sqrt{2x+1}-3}{x^3-64} &= \lim_{x\to 4}\frac{\sqrt{2x+1}-3}{(x-4)\left(x^2+4x+16\right)}\cdot\frac{\sqrt{2x+1}+3}{\sqrt{2x+1}+3}\\
&= \lim_{x\to 4}\frac{2x-8}{(x-4)\left(x^2+4x+16\right)\left(\sqrt{2x+1}+3\right)}\\
&= \lim_{x\to 4}\frac{2}{\left(x^2+4x+16\right)\left(\sqrt{2x+1}+3\right)}\\
&= \frac{2}{\left(4^2+4^2+4^2\right)\left(\sqrt{9}+3\right)}=\frac{1}{144}
\end{aligned}$$

(b)

Solution

$$\begin{aligned}
\lim_{x\to -\infty}\frac{\left(x^4+1\right)(3-2x)^3}{(x+1)^5\left(2-x^2\right)} &= \lim_{x\to -\infty}\frac{\left[\left(x^4+1\right)/x^4\right]\left[(3-2x)^3/x^3\right]}{\left[(x+1)^5/x^5\right]\left[\left(2-x^2\right)/x^2\right]}\\
&= \lim_{x\to -\infty}\frac{\left(x^4/x^4+1/x^4\right)(3/x-2x/x)^3}{(x/x+1/x)^5\left(2/x^2-x^2/x\right)}\\
&= \lim_{x\to -\infty}\frac{\left(1+1/x^4\right)(3/x-2)^3}{(1+1/x)^5\left(2/x^2-1\right)}=\frac{1\cdot(-2)^3}{1\cdot(-1)}\\
&= 8
\end{aligned}$$

3. (a)

Solution

$$f(x)=\frac{|x+1|}{x^2-1}=\frac{|x+1|}{(x-1)(x+1)}$$

is undefined only at $x = 1$ and -1, and we see that

$$\begin{aligned}
\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} \frac{|x+1|}{(x+1)} \lim_{x\to 1^-} \frac{1}{(x-1)} \\
&= (+1)\cdot(-\infty) = -\infty \\
\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} \frac{|x+1|}{(x+1)} \lim_{x\to 1^+} \frac{1}{(x-1)} \\
&= (+1)\cdot(+\infty) = +\infty \\
\lim_{x\to -1^-} f(x) &= \lim_{x\to 1^-} \frac{|x+1|}{(x+1)} \lim_{x\to -1^-} \frac{1}{(x-1)} \\
&= (-1)\cdot\left(-\frac{1}{2}\right) = \frac{1}{2} \\
\lim_{x\to -1^+} f(x) &= \lim_{x\to 1^+} \frac{|x+1|}{(x+1)} \lim_{x\to -1^+} \frac{1}{(x-1)} \\
&= (+1)\cdot\left(-\frac{1}{2}\right) = -\frac{1}{2}
\end{aligned}$$

(b)

Solution

$$\begin{aligned}
f(x) &= x+5 \text{ if } x<0 \text{ and} \\
f(x) &= (x-a)^2 + b \text{ if } 0 \le x < 2 \text{ and so} \\
\lim_{x\to 0^-} f(x) &= 5 = \lim_{x\to 0^+} f(x) = f(0) = a^2 + b
\end{aligned}$$

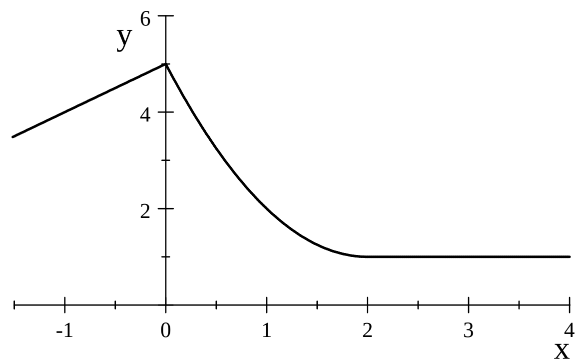
for continuity. Similarly,

$$\lim_{x\to 2^-} f(x) = 1 = 4 - 4a + a^2 + b$$

So we have the two equations

$$\begin{aligned}
a^2 + b &= 5 \\
a^2 + b - 4a &= -3
\end{aligned}$$

Subtracting gives $4a = 8$ and so $a = 2$ and finally $b = 1$. The graph is below.



4. (a) $f(x) = \dfrac{\sqrt[3]{x} - 2\sqrt[5]{x^3} + x^4}{\sqrt{x}} = \dfrac{x^{1/3} - 2x^{3/5} + x^4}{x^{1/2}} = x^{-1/6} - 2x^{1/10} + x^{3.5}$

Solution $f'(x) = -\dfrac{1}{6}x^{-7/6} - \dfrac{2}{10}x^{-9/10} + 3.5x^{2.5}$ (of course if you don't simplify before differentiating the answer will look different).

(b) $f(x) = e^{-2x^3}(\sin x + \tan x)^2$

Solution

$f'(x) = e^{-2x^3} \cdot 2(\sin x + \tan x)(\cos x + \sec^2 x) + (-6x^2)e^{-2x^3}(\sin x + \tan x)^2$ (product, chain rules)

(c) $f(x) = \sec(\arcsin 2x)$

Solution

$$f'(x) = \sec(\arcsin 2x)\tan(\arcsin 2x) \cdot \frac{1}{\sqrt{1-4x^2}} \cdot 2 \text{ (chain rule)}$$

(d) $f(x) = \dfrac{\ln^2(\sqrt{x})}{1+\sqrt{e^{2x}}} = \dfrac{\ln^2(x)}{4(1+e^x)}$ because $\ln(\sqrt{x}) = \frac{1}{2}\ln(x)$

Solution $f'(x) = \dfrac{1}{4}\dfrac{(1+e^x) \cdot 2\ln x \cdot 1/x - \ln^2(x) \cdot e^x}{(1+e^{x/2})^2}$

(e) $f(x) = (\arctan(x))^{1+x^2}$

Solution We take logarithms first:

$$\begin{aligned}\ln f &= \left(1+x^2\right)\cdot \ln\left(\arctan\left(x\right)\right)\\ \frac{f'}{f} &= 2x\ln\left(\arctan\left(x\right)\right)+\left(1+x^2\right)\cdot\frac{1}{\arctan\left(x\right)}\frac{1}{1+x^2}\\ f'\left(x\right) &= \left(\arctan\left(x\right)\right)^{1+x^2}\left(2x\ln\left(\arctan\left(x\right)\right)+\frac{1}{\arctan\left(x\right)}\right)\end{aligned}$$

5. (a) $f\left(x\right)=x^2+\dfrac{1}{x}$

Solution

$$\begin{aligned}\frac{f\left(x+h\right)-f\left(x\right)}{h} &= \frac{\left(x+h\right)^2+\frac{1}{x+h}-x^2-\dfrac{1}{x}}{h}\\ &= \frac{2xh+h^2}{h}+\frac{x-\left(x+h\right)}{h\cdot x\cdot\left(x+h\right)}\\ &= 2x+h-\frac{1}{x\cdot\left(x+h\right)};h\neq 0\end{aligned}$$

and so

$$\begin{aligned}f'\left(x\right) &= \lim_{h\to 0}\frac{f\left(x+h\right)-f\left(x\right)}{h}\\ &= \lim_{h\to 0}2x+h-\frac{1}{x\cdot\left(x+h\right)}\\ &= 2x-\frac{1}{x^2}\end{aligned}$$

(b)

Solution Since $f\left(x\right)=x^2+\dfrac{1}{x}=x^2+x^{-1}$ the power rule gives $f'\left(x\right)=2x-x^{-2}$ which is the same as above.

6. (a) $f\left(x\right)=\tan x$ and $a=\pi/4$

Solution

$$\begin{aligned}df &= f'\left(a\right)dx\\ &= \sec^2\left(\pi/4\right)dx\\ &= 2dx \text{ since } \sec^2\left(\pi/4\right)=2\end{aligned}$$

(b) Estimate $\tan 0.8$ using the above differential.

Solution

Assume $\pi/4 = .785$. Then

$$\begin{aligned} dx &= .8 - .785 = .015 \\ df &= 2dx = .030 \end{aligned}$$

Since $\tan \pi/4 = 1$ it follows that $\tan 0.8 \approx 1 + 0.030 = 1.030$ (note that the exact value is 1.0296

7.

$$y^2\left(y^2 - 4\right) = x^2\left(x^2 - 5\right)$$

(a)

Solution $4(4-4) = 0$ so the point $(0, -2)$ is on the curve. By implicit differentiation we have

$$\begin{aligned} 2yy'\left(y^2 - 4\right) + y^2(2yy') &= 2x\left(x^2 - 5\right) + x^2(2x) \text{ now substitute} \\ 0 + 4 \cdot (-4)\, y' &= 0 \\ y' &= 0 \end{aligned}$$

So the tangent line has equation $y - (-2) = 0$ or $y = -2$

(b)

Solution Since $\sin 0 - 0 = 0 = 0 \cos 0 - 0$ we have an indeterminate form of the $\frac{0}{0}$ type so L'Hopital's rule applies:

$$\begin{aligned} \lim_{x \to 0} \frac{\sin x - x}{x \cos x - x} &= \lim_{x \to 0} \frac{\cos x - 1}{\cos x - x \sin x - 1}; \text{ still } \frac{0}{0} \text{ type} \\ &= \lim_{x \to 0} \frac{-\sin x}{-\sin x - \sin x - x \cos x} \\ &= \lim_{x \to 0} \frac{\sin x}{2 \sin x + x \cos x}; \text{ still } \frac{0}{0} \text{ type} \\ &= \lim_{x \to 0} \frac{\cos x}{2 \cos x + \cos x - x \sin x} = \frac{1}{3} \end{aligned}$$

8. (a) $f(x) = e^{\ln(\arctan(2x))}$

Solution Simplifying first gives

$$\begin{aligned} f(x) &= \arctan(2x) \text{ and so} \\ f'(x) &= \frac{2}{1+4x^2} \\ f''(x) &= \frac{(1+4x^2)\cdot 0 - 2\cdot 8x}{(1+4x^2)^2} = \frac{-16x}{(1+4x^2)^2} \end{aligned}$$

(b) Let $f(x) = x^3 + x - 1$; find c satisfying MVT on $[0,2]$.

Solution

$$\begin{aligned} f(0) &= -1;\ f(2) = 9 \\ \frac{f(2) - f(0)}{2-0} &= \frac{9+1}{2} = 5 \\ f'(x) &= 3x^2 + 1 \text{ so we want} \\ f'(c) &= 3c^2 + 1 = 5 \\ c^2 &= \frac{4}{3} \\ c &= \sqrt{\frac{4}{3}} \end{aligned}$$

9. (a) At what rate is the area of an equilateral triangle increasing if its base is

10 cm long and increasing at 0.5 cm/s?

Solution Let x be the length of a side of the triangle. The area (call it A) is then given by

$$A = \frac{1}{2}x \cdot h$$

where h is the altitude. But it is clear that $\frac{h}{x} = \sin(\pi/3) = \sqrt{3}/2$ for an equilateral triangle (with all angles equal to $\pi/3$). So

$$\begin{aligned} A &= \frac{\sqrt{3}}{4}x^2; \text{ now differentiate with respect to } t \\ \frac{dA}{dt} &= \frac{\sqrt{3}}{2}x\frac{dx}{dt} \\ &= \frac{\sqrt{3}}{2}(10)(0.5) = \frac{5\sqrt{3}}{2} \text{ cm}^2/\text{s} \end{aligned}$$

or approximately 4.33 cm^2/s.

(b) If 1200 cm^2 of material is available to make a box with a square base and

no top, find the largest possible volume of the box.

Solution Let x be the length of the base, h the height and V the total volume. We are given that

$$\begin{aligned} x^2 + 4xh &= 1200 \text{ and want to maximize} \\ V &= x^2 h \end{aligned}$$

So we eliminate h using the first equation: $h = \dfrac{1}{4x}\left(1200 - x^2\right)$ and so

$$\begin{aligned} V &= \frac{x}{4}\left(1200 - x^2\right) = -\frac{x^3}{4} + 300x \\ \frac{dV}{dx} &= -\frac{3x^2}{4} + 300 = 0 \Longrightarrow x = 20 \text{ and } h = \frac{1200}{400} = 3 \end{aligned}$$

To check that we have a maximum,

$$\frac{d^2V}{dx^2} = -\frac{3x}{2} + 300 = -30 + 300 < 0$$

so we do have a local and hence an absolute maximum.

10.

$$f(x) = \frac{2x^2}{x^2 - 1}$$

Solution

(a) Find the domain and check for symmetry. Find all asymptotes.

Solution The domain consists of all numbers for which the formula is defined, which means for which the denominator $\neq 0$. So $x^2 - 1 = (x-1)(x+1) \neq 0$ which means $x \neq 1$ and $x \neq -1$. The domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ and there are vertical asymptotoes at $x = \pm 1$ since the function has infinite one-sided limits at those points. Also,

$$\lim_{x \to \infty} \frac{2x^2}{x^2 - 1} = 2$$

so $y = 2$ is a horizontal asymptote. Since $f(-x) = f(x)$ the function is even (even symmetry).

(b) Calculate $f'(x)$ and use it to determine interval(s) where the function is

increasing, interval(s) where the function is decreasing, and local extrema

(if any).

Solution

$$\begin{aligned} f'(x) &= 4\frac{x}{x^2-1} - 4\frac{x^3}{(x^2-1)^2} \\ &= -4\frac{x}{(x-1)^2(x+1)^2} \end{aligned}$$

so $x = 0$ is a critical point. We have the following table (valid except for the points ± 1, which are not in the domain):

	$x < 0$	$x = 0$	$x > 0$
$f'(x)$	$+$	0	$-$
f	$\nearrow$	l. max	$\searrow$

(c) Calculate $f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

Solution

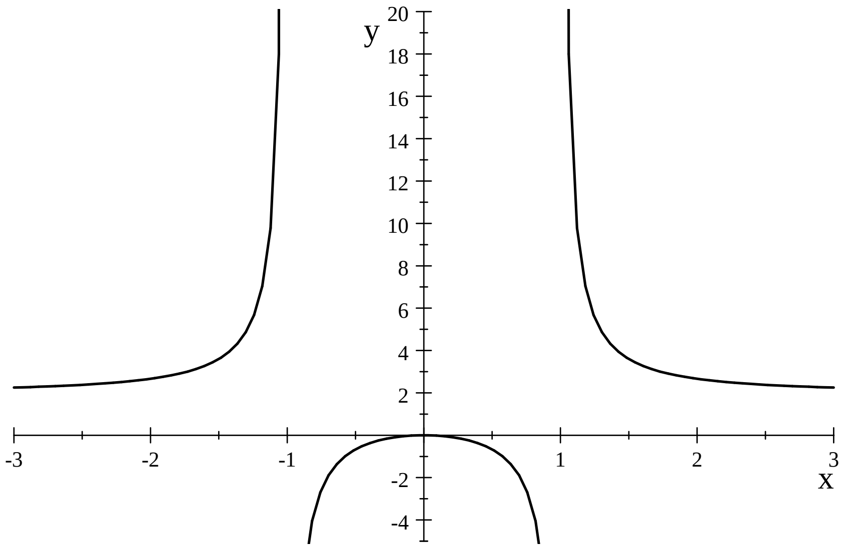
$$f''(x) = 4\frac{3x^2+1}{(x-1)^3(x+1)^3}$$

We see that there are no possible inflection points, but the concavity does change as we cross the vertical asymptotes. Here is the analysis:

	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$x - 1$	$-$		$-$		$+$
$x + 1$	$-$		$+$		$+$
$f''(x)$	$+$	undef.	$-$	undef.	$+$
	$\cup$		$\cap$		$\cup$

(d) Sketch the graph of the function.

Solution



Bonus Let $g(t)$ be the distance the first runner covers in time t and let $h(t)$ be the distance the second runner covers in time t. We know $g(0) = h(0)$ and also $g(T) = h(T)$ where T is the time is takes either one to run the race.Put $f(t) = g(t) - h(t)$. Now $f(0) = f(T) = 0$ and so, by Rolles' Theorem, there must be a time t_c at which

$$\begin{aligned} \frac{f(T) - f(0)}{T - 0} &= f'(t_c), \text{ i.e.} \\ 0 &= f'(t_c) = g'(t_c) - h'(t_c) \text{ and so} \\ g'(t_c) &= h'(t_c) \end{aligned}$$

Which means the speed of the two runners at time t_c is exactly the same.