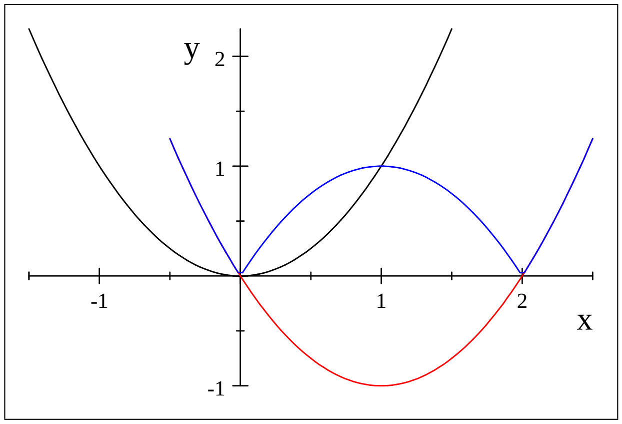
MATH 203 Final Exam December 2007
Solutions

1. (a) Sketch the graph of the function $f(x) = |(x-1)^2 - 1|$, starting from the graph

of the standard parabola and using appropriate transformations.

Solution



(b) Suppose $f(x) = \dfrac{2x+1}{x+1}$ and $g(x) = \dfrac{x-1}{2-x}$. Find and $g \circ f$

Solution $f \circ g\,(x) = \dfrac{2\dfrac{x-1}{2-x} + 1}{\dfrac{x-1}{2-x} + 1} = \dfrac{2x - 2 + 2 - x}{x - 1 + 2 - x} = x$ and $g \circ f\,(x) =$

$$\frac{\dfrac{2x+1}{x+1} - 1}{2 - \dfrac{2x+1}{x+1}} = \frac{2x + 1 - x - 1}{2x + 2 - 2x - 1} = x$$

(c) Solve for x:

$$3^{\log_3 x^2} = 2e^{\ln x} + 4 \cdot 10^{\log_{10} 2}$$

Solution

$$\begin{aligned}
x^2 &= 2x + 4 \cdot 2 \\
x^2 - 2x - 8 &= 0 \\
(x-4)(x+2) &= 0 \\
x &= 4
\end{aligned}$$

because x cannot be negative in the expression $e^{\ln x}$.

2. (a)

Solution

$$\begin{aligned}\lim_{t\to -1}\frac{t^2+3t+2}{t^2-t-2} &= \lim_{t\to -1}\frac{(t+1)(t+2)}{(t+1)(t-2)}\\ &= \lim_{t\to -1}\frac{(t+2)}{(t-2)}\\ &= \frac{\lim_{t\to -1}(t+2)}{\lim_{t\to -1}(t-2)} = \frac{1}{-3} = -\frac{1}{3}\end{aligned}$$

(b)

Solution

$$\begin{aligned}\lim_{x\to 9}\frac{9x-x^2}{3-\sqrt{x}} &= \lim_{x\to 9}\frac{x(9-x)}{3-\sqrt{x}}\cdot\frac{3+\sqrt{x}}{3+\sqrt{x}}\\ &= \lim_{x\to 9}\frac{x(9-x)}{9-x}\cdot\left(3+\sqrt{x}\right)\\ &= \lim_{x\to 9}x\cdot\left(3+\sqrt{x}\right)\\ &= 9\cdot(3+3) = 54\end{aligned}$$

(c)

Solution

$$\begin{aligned}\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}} &= \lim_{x\to\infty}\frac{1-e^{-2x}}{1+e^{-2x}}\\ &= \frac{1}{1} = 1\end{aligned}$$

3. (a)

Solution

$$\begin{aligned}f(x) &= \frac{x^2-x-6}{|x-3|} = \frac{(x+2)(x-3)}{|x-3|}\\ &= \frac{(x-3)}{|x-3|}\cdot(x+2)\end{aligned}$$

is undefined only at $x = 3$ and we see that

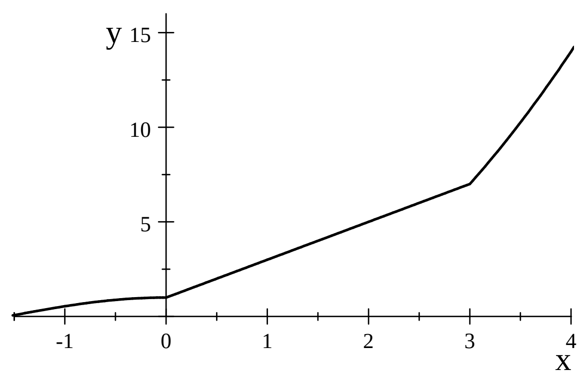
$$\begin{aligned}
\lim_{x\to 3^-} f(x) &= \lim_{x\to 3^-} \frac{(x-3)}{|x-3|} \lim_{x\to 3^-} (x+2) \\
&= (-1)\cdot(3+2) = -5 \\
\lim_{x\to 3^+} f(x) &= \lim_{x\to 3^+} \frac{(x-3)}{|x-3|} \lim_{x\to 3^+} (x+2) \\
&= (+1)\cdot(3+2) = +5
\end{aligned}$$

(b)

Solution

$$\begin{aligned}
f(x) &= \cos x \text{ if } x < 0 \text{ and} \\
f(x) &= ax + b \text{ if } 0 \le x < 3 \text{ and so} \\
\lim_{x\to 0^-} f(x) &= 1 = \lim_{x\to 0^+} f(x) = f(0) = b
\end{aligned}$$

for continuity. So $b = 1$. Similarly, $\lim_{x\to 3^-} f(x) = 3a + 1 = \lim_{x\to 3^+} f(x) = 3^2 - 2 = f(3)$ and so $3a + 1 = 7 \Longrightarrow a = 2$.



4. (a) $f(x) = (x + x^{-1})^2 \cos 2x$

Solution $f'(x) = 2(x + x^{-1})(1 - x^{-2})\cos 2x - 2(\sin 2x)(x + x^{-1})^2$

(b) $f(x) = \dfrac{\sin^{-1}\left(\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}$

$$f'(x) = \frac{\sqrt{1-x^2} \cdot \frac{1}{\sqrt{1-(1-x^2)}} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) - \sin^{-1}\left(\sqrt{1-x^2}\right) \cdot \frac{1}{2\sqrt{1-x^2}} \cdot (-2x)}{1-x^2} =$$

$$-\frac{x}{(1-x^2)\sqrt{x^2}} + x\frac{\sin^{-1}\sqrt{1-x^2}}{(1-x^2)^{\frac{3}{2}}}$$

(c) $f(x) = (x^2)^\pi + \pi^{x^2} \quad = x^{2\pi} + \pi^{x^2}$

$f'(x) = 2\pi x^{2\pi-1} + \pi^{x^2} \cdot \ln \pi \cdot 2x$

(d) $f(x) = 4x\sqrt{x+\sqrt{x}}$

Solution $f'(x) = 4\sqrt{x+\sqrt{x}} + 2\frac{x}{\sqrt{x+\sqrt{x}}}\left(1 + \frac{1}{2\sqrt{x}}\right)$

(e) $f(x) = \left(\tan^{-1}(2x)\right)^{\ln x}$

Solution We take logarithms first:

$$\begin{aligned}
\ln f &= \ln x \cdot \ln\left(\tan^{-1}(2x)\right) \\
\frac{f'}{f} &= \frac{1}{x}\ln\left(\tan^{-1}(2x)\right) + \ln x \cdot \frac{1}{\tan^{-1}(2x)}\frac{1}{1+4x^2} \cdot 2 \\
f'(x) &= \left(\tan^{-1}(2x)\right)^{\ln x}\left(\frac{1}{x}\ln\left(\tan^{-1}(2x)\right) + \frac{1}{\tan^{-1}(2x)}\frac{2\ln x}{1+4x^2}\right)
\end{aligned}$$

5. (a) $f(x) = (1+x)^n$

Solution $f'(x) = n(1+x)^{n-1}$ and $f'(0) = n$ so $L(x) = f(0) + xf'(0) = 1 + nx$. For $n = 50$ and $x_0 = .003$ we get $L(.003) = 1 + 50(.003) = 1.15$. (We note that the actual value of 1.003^{50} is 1.162)

(b)

Solution Write

$$\begin{aligned}
df &= f'(a)\,dx \quad \text{where } a = 0 \text{ and } dx = .003, \text{ so} \\
df &= f'(0)(.003) \\
&= (50)(.003) = 0.15 \\
\text{So } f(0+dx) &\approx f(0) + df = 1 + 0.15 = 1.15
\end{aligned}$$

(c) $g(x) = (x-1)^2$

Solution

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{(x+h-1)^2-(x-1)^2}{h} \\ &= \lim_{h\to 0}\frac{x^2+2xh+h^2-2x-2h+1-x^2+2x-1}{h} \\ &= \lim_{h\to 0}\frac{2xh+h^2-2h}{h} = \lim_{h\to 0}\frac{h}{h}(2x+h-2) = 2x-2 \end{aligned}$$

(d) $g(x) = (x-1)^2$

Solution

$$\begin{aligned} g'(x) &= 2(x-1)\cdot 1 \text{ (chain rule)} \\ &= 2x-2 \end{aligned}$$

6. $x - 2y^2 + 5 = 3e^{x/y}$

(a)

Solution

$$\begin{aligned} 0-2\cdot 1+5 &= 3e^0 \\ 3 &= 3 \end{aligned}$$

so the point $(0,1)$ is on the curve. Now differentiate:

$$\begin{aligned} 1-4yy' &= 3e^{x/y}\left(1/y - x/y^2y'\right) \text{ now substitute } (0,1) \\ 1-4y' &= 3(1-0) \\ y' &= -\frac{1}{2} \text{ and a tangent line is} \\ y-1 &= -\frac{1}{2}(x-0) \text{ or} \\ y &= -\frac{1}{2}x+1 \end{aligned}$$

(b) $f(x) = x^2 + 2x - 1$ on $[0,1]$.

Solution $f'(x) = 2x+2$ and so we need

$$\begin{aligned} f'(c) &= \frac{f(1)-f(0)}{1-0} = \frac{2-(-1)}{1} = 3 \\ 2c+2 &= 3 \\ 2c &= 1 \end{aligned}$$

So the solution is $c = \frac{1}{2}$.

(c)

$$\lim_{x\to 0}\frac{\tan x - x}{x^3}$$

Solution Check: a $\frac{0}{0}$ type expression, so L'Hopital's rule applies

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan x - x}{x^3} &= \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2} \text{ repeat} \\
&= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x} \text{ repeat again} \\
&= \lim_{x\to 0}\frac{4\sec^2 x\tan^2 x + 2\sec^4 x}{6} = \frac{2}{6} \\
&= \frac{1}{3}
\end{aligned}$$

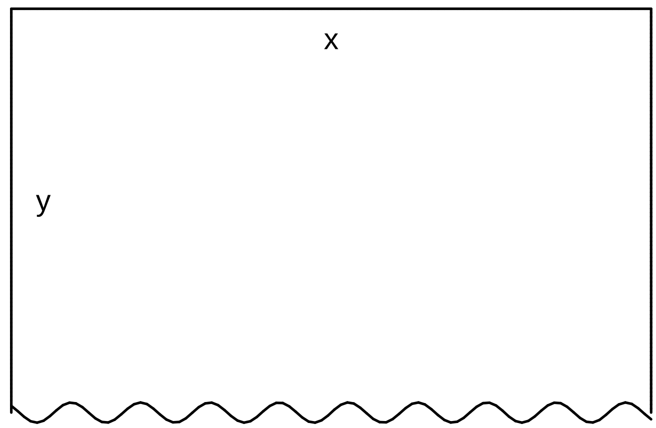
7. (a)

Solution We have the relationship

$$\begin{aligned}
s^2 &= x^2 + h^2 \text{ and also at time } t_0 \\
s^2 &= 40^2 + 4^2 = 1616 \Longrightarrow s = 40.2 \\
\text{Differentiate} & \quad \text{with respect to time } t \text{ to get} \\
2ss'(t) &= 2xx'(t) \text{ since } h \text{ is constant} \\
x'(t) &= \frac{s(t)\,s'(t)}{x(t)} \text{ and at } t_0 \text{ we get} \\
x'(t_0) &= \frac{40.2\cdot(-400)}{40} = -402 \text{ km/h}
\end{aligned}$$

(b)

Solution Draw a sketch:



We want to maximize

$$A = xy$$

subject to the condition that

$$x + 2y = 800$$

So we eliminate y by expressing it in terms of $x : y = \frac{1}{2}(800 - x)$ and substitute to get

$$\begin{aligned} A &= \frac{1}{2}x(800 - x) = 400x - \frac{1}{2}x^2 \\ A' &= 400 - x = 0 \text{ for a critical point, so} \\ x &= 400 \text{ and consequently } y = 200 \end{aligned}$$

To check that this is the maximum area, we calculate $A'' = -1 < 0$, which means the graph is concave down and it is indeed a maximum.

8. (a) Find the domain and check for symmetry. Find asymptotes (if any).

Solution

$$y = f(x) = 4x^3 - x^4$$

Since this is a polynomial with odd and even powers, there is no symmetry and no asymptotes.

(b) Calculate $f'(x)$ and use it to determine interval(s) where the function is increasing, interval(s) where the function is decreasing, and local extrema (if any).

Solution

$$f'(x) = 12x^2 - 4x^3 = 4x^2(3 - x)$$

We see that $x = 0, 3$ are critical points. Since $x^2 \geq 0$, the sign of $f'(x)$ changes only when x passes through $x = 3$. Here is what is happening.

	$x < 3$	$x = 3$	$x > 3$
	$+$	0	$-$
$f'(x)$	$\nearrow$	l. max	$\searrow$

So there is a local maximum at $(3, 27)$ which is also an absolute maximum.

(c) Calculate $f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

Solution

$$f''(x) = 24x - 12x^2 = 12x(2 - x)$$

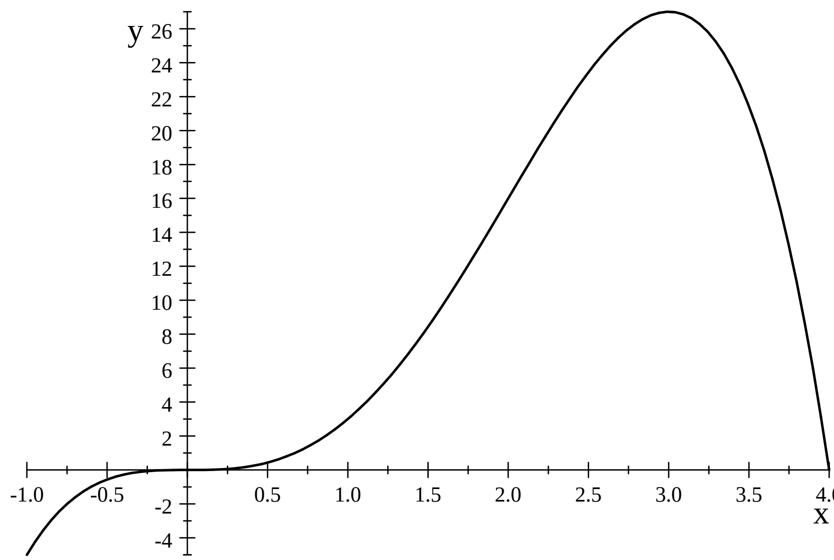
We see that $x = 0, 2$ are possible inflection points. Here is the analysis.

	$x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
$12x$	$-$	0	$+$	$+$	$+$
$2 - x$	$+$	$+$	$+$	0	$-$
$f''(x)$	$\cap$	i.p.	$\cup$	i.p.	$\cap$

So there are inflection points at $(0, 0)$ and $(2, 16)$

(d) Sketch the graph of the function.

Solution



Bonus

$$f(x) = \begin{cases} -x^3 & \text{if} \quad x \geq 0 \\ x^3 & \text{if} \quad x < 0 \end{cases}$$

Also, if $x > 0$ then $f'(x) = -3x^2$. If $x < 0$ then $f'(x) = 3x^2$. So the only question is what happens at $x = 0$. We can't just substitute $x = 0$ into the equation and get the answer 0, because we don't know ahead of time that $f'(0)$ even exists!. What we have to do is go back to the definition:

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\pm h^3 - 0}{h} \\ &= \lim_{h \to 0} \pm h^2 = 0 \end{aligned}$$

where we put $\pm$ depending on whether $h > 0$ (it would be $-$) or $h < 0$ (it would be $+$). Either way, the limit is zero, so the derivative does indeed exist at 0 (and everywhere).