9.2 a) If $\rho = 1$ then $Z_1 = Z_2$, so

$$Y_1 = X + Z_1$$
$$Y_2 = X + Z_1 = Y_1$$

So Y_2 provides no additional information over Y_1.

$$\Rightarrow \quad C = \frac{1}{2} \log\left(1 + \frac{P}{N}\right)$$

c) Here $Z_2 = -Z_1$

$$\Rightarrow \quad Y_1 = X + Z_1$$
$$Y_2 = X - Z_1$$

$$\Rightarrow \quad \frac{Y_1 + Y_2}{2} = X,$$ a noise free observation of X

$$\Rightarrow \quad C = \infty$$

9.3 $C = \max\limits_{p_x : EY^2 \le P} I(X;Y)$

$$I(X;Y) = h(Y) - h(\cancel{Y}|X)$$
$$\le \frac{1}{2}\log 2\pi e P - h(Y|X)$$
$$= \frac{1}{2}\log 2\pi e P - h(Z)$$
$$= \frac{1}{2}\log 2\pi e P - \frac{1}{2}\log(2\pi e \sigma^2)$$
$$= \frac{1}{2}\log P/\sigma^2$$

we have equality when $Y \sim N(0, P)$
which happens when $X \sim N(0, P - \sigma^2)$

9.5 $\quad Y = XV + Z$

$$I(X;Y) \le I(X;Y,V)$$

$$= \underbrace{I(X;V)}_{=0} + I(X;Y|V)$$

$$= I(X;Y|V)$$

where $I(X;V) = 0$ since X & V are independant

9.6 Recall that

$$P_1 = EX_1^2$$
$$P_2 = EX_2^2$$

and $P_j = [\nu - \sigma_j^2]^+ \qquad j = 1, 2$

We are at the border of assigning power to channel 1 when $\nu = \sigma_1^2$

$\Rightarrow \quad P_2 = \sigma_1^2 - \sigma_2^2$ and $P_1 = \sigma_1^2 - \sigma_1^2 = 0$

$\Rightarrow \quad EX_1^2 + EX_2^2 = \sigma_1^2 - \sigma_2^2$

9.7

a) $Y = (X + Z_1) + (X + Z_2)$

$= 2X + Z_1 + Z_2$

Signal power: $E(2X)^2 = 4P^2$

Noise Power: $E(Z_1 + Z_2)^2 = \sigma^2 + 2\rho\sigma^2 + \sigma^2$

$\Rightarrow C = \frac{1}{2}\log\left(1 + \frac{4P}{2\sigma^2 + 2\rho\sigma^2}\right)$

$= \frac{1}{2}\log\left(1 + \frac{2P}{\sigma^2(1+\rho)}\right)$

b) $\rho = 0 \Rightarrow C = \frac{1}{2}\log\left(1 + \frac{2P}{\sigma^2}\right)$

$\rho = 1 \Rightarrow C = \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right)$

$\rho = -1 \Rightarrow C = \infty$

9.12

$$nR = H(W) = I(W;\hat{W}) + H(W|\hat{W})$$

$$\le I(W;\hat{W}) + n\varepsilon_n$$

$$\le I(X_1^n; Y_1^n) + n\varepsilon_n$$

$$= h(Y_1^n) - h(Y_1^n|X_1^n) + n\varepsilon_n$$

$$\le \sum_{\ell=1}^{n} h(Y_\ell) - h(Z_\ell) + n\varepsilon_n$$

$$= \sum_{\ell=1}^{n} I(X_\ell; Y_\ell) + n\varepsilon_n$$

Now, $\frac{1}{n}\sum_{\ell=1}^{n} x_\ell^2(w) \le P$

$$\Rightarrow \frac{1}{n}\sum_{\ell=1}^{n} E[X_\ell^2] \le P$$

$$\Rightarrow E[X_\ell^2] \le nP \quad \text{for } \ell = 1, \ldots, n$$

$$\Rightarrow E\left[\left(\frac{X_\ell}{\ell}\right)^2\right] \le \frac{nP}{\ell^2}$$

$$\Rightarrow nR \le \sum_{\ell=1}^{n} \frac{1}{2}\log\left(1 + \frac{nP}{\ell^2 N}\right) + n\varepsilon_n$$

$$R \le \frac{1}{2}\cdot\frac{1}{n}\sum_{\ell=1}^{n} \log\left(1 + \frac{nP}{\ell^2 N}\right) + \varepsilon_n$$

Now, $\epsilon_n \to 0$ as $n \to \infty$, and

$$\frac{1}{n} \sum_{\ell=1}^{n} \log\left(1 + \frac{nP}{\ell^2 N}\right)$$

$$\leq \frac{1}{n} \sum_{\ell=1}^{\infty} \log\left(1 + \frac{nP}{\ell^2 N}\right)$$

$$\to 0 \quad \text{as} \quad n \to \infty.$$

This is because

$$\lim_{n\to\infty} \frac{\sum_{\ell=1}^{\infty} \log\left(1 + \frac{nP}{\ell^2 N}\right)}{n}$$

$$= \lim_{n\to\infty} \frac{\left[\sum_{\ell=1}^{\infty} \frac{1}{1 + \frac{nP}{\ell^2 N}} \cdot \frac{P}{\ell^2 N}\right]}{1}$$

by l'Hôpital's rule and this limit is 0.

So $R \leq 0$

9.14 $\quad C \geq I(X;Y)$ for any $X \sim p_X$.

Let $X \sim N(0, P)$. Then $h(Y)$ is finite.

Now $f_Z(z) = \frac{1}{10}\delta(z) + \frac{9}{10} g_{Z^*}(z)$

where $g_{Z^*}(z)$ is pdf of $Z^* \sim N(0, N)$

Then
$$h(Z) = -\int f_Z(z) \log f_Z(z)\, dz$$
$$\leq -\int \frac{1}{10}\delta(z) \log\left[\frac{1}{10}\delta(z)\right] dz$$
$$= -\infty$$

So $C \geq h(Y) - [-\infty] = \infty$.

The following strategy works:

Let $B_1, B_2, B_3, \ldots$ be the infinite sequence of bits to transmit in a finite time n.

Construct the random number $T = \sum_{\ell=1}^{\infty} B_\ell / 2^\ell$

Pick $X = \frac{\sqrt{P}}{E[T^2]} \cdot T$

Then $E X^2 = P$

Also, from T (or equivalently X), we can recover the infinite # of bits error free.

To send X ~~almost~~ error free with high probability, choose the inputs $X_1 = X_2 = \cdots = X_n = X$.

The receiver looks for 2 Y's, such that

$Y_i = Y_j$ for $i \neq j$.

If this happens it is because $Y_i = Y_j = X$, and the receiver knows X now.

The probability of this <u>not</u> happening decays exponentially to 0 as $n \to \infty$.