Techniques of Integration

MAT1322 C Winter 2020

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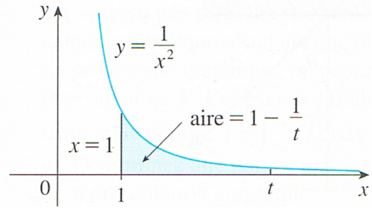
Plan

1. Improper Integrals
    - Type 1: Infinite Intervals
    - Type 2: Discontinuous Integrands
    - A Comparison Test for Improper Integrals

Type 1: Infinite Intervals

- Consider the infinite region S that lies under the curve $y = 1/x^2$, above the x-axis, and to the right of the line $x = 1$.



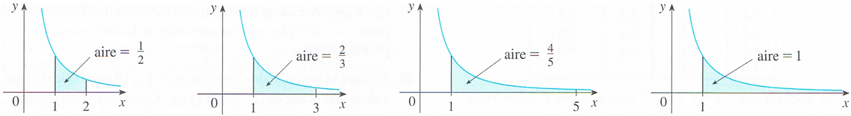
- The area of the part of S that lies to the left of the line $x = t$ (shaded in the figure above) is

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^t = 1 - \frac{1}{t}.$$

- We also observe that

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1-\frac{1}{t}\right) = 1.$$

- The area of the shaded region approaches 1 as $t \to \infty$



- So we say that the area of the infinite region S is equal to 1 and we write

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^{t}\frac{1}{x^2}\,dx = 1.$$

Definition (Definition of an Improper Integral of Type 1)

(a) *If* $\int_a^t f(x)dx$ *exists for every number* $t \geq a$, *then*

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\left(\int_a^t f(x)dx\right)$$

provided this limit exists (as a finite number).

(b) *If* $\int_t^b f(x)dx$ *exists for every number* $t \leq b$, *then*

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\left(\int_t^b f(x)dx\right)$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)dx$ *and* $\int_{-\infty}^b f(x)dx$ *are called* **convergent** *if the corresponding limit exists and* **divergent** *if the limit does not exist.*

(c) *If both* $\int_a^\infty f(x)dx$ *and* $\int_{-\infty}^a f(x)dx$ *are convergent, then we define*

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

In part (c) any real number a *can be used.*

Example

Determine whether the integral

$$\int_1^{\infty} \frac{1}{x}\, dx$$

is convergent or divergent.

Solution:

- According to part (a) of the above definition, we have

$$\begin{aligned}\int_1^{\infty} \frac{1}{x}\,dx &= \lim_{t\to\infty}\int_1^{t}\frac{1}{x}\,dx\\ &= \lim_{t\to\infty}\left[\ln|x|\right]_1^t\\ &= \lim_{t\to\infty}(\ln t - \ln 1)\\ &= \infty\end{aligned}$$

- The limit does not exist as a finite number and so the improper integral

$$\int_1^{\infty}\frac{1}{x}\,dx$$

is divergent.

Example

Evaluate

$$\int_{-\infty}^{0} xe^{x}\, dx.$$

Solution:

- Using part (b) of the previous definition, we have

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx.$$

- We integrate by parts with $u = x$ and $dv = e^x\,dx$, and so $du = dx$ and $v = e^x$:

$$\int_t^0 xe^x\,dx = [xe^x]_t^0 - \int_t^0 e^x\,dx = -te^t - 1 + e^t.$$

- We know that $e^t \to 0$ as $t \to -\infty$, and by l'Hospital's Rule we have

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\frac{t}{e^{-t}} = \lim_{t\to-\infty}\frac{1}{-e^{-t}} = \lim_{t\to-\infty}(-e^{-t}) = 0.$$

- Therefore

$$\begin{aligned}\int_{-\infty}^{0} xe^x\,dx &= \lim_{t\to-\infty}\left(-te^t - 1 + e^t\right)\\ &= -0 - 1 + 0\\ &= -1.\end{aligned}$$

Example

Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$$

Solution:

- It's convenient to choose $a = 0$ in part (c) of our previous definition:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx.$$

- We must now evaluate the integrals on the right side separately:

$$\begin{aligned}
\int_{0}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{t\to\infty} \int_0^t \frac{dx}{1+x^2} \\
&= \lim_{t\to\infty} [\arctan x]_0^t \\
&= \lim_{t\to\infty} (\arctan t - \arctan 0) \\
&= \frac{\pi}{2}
\end{aligned}$$

Solution (continued):

- (continued)

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{t\to-\infty}\int_t^0 \frac{dx}{1+x^2} \\ &= \lim_{t\to-\infty}\left[\arctan x\right]_t^0 \\ &= \lim_{t\to-\infty}\left(\arctan 0 - \arctan t\right) \\ &= \frac{\pi}{2}\end{aligned}$$

- Since both of these integrals are convergent, the given integral is convergent and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Example

For what values of p is the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

convergent?

Solution:

- We know from our first example that if $p = 1$, then the integral is divergent, so let's assume $p \neq 1$.
- Then

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{t\to\infty}\int_1^t x^{-p}\,dx \\ &= \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t \\ &= \lim_{t\to\infty}\frac{1}{1-p}\left(\frac{1}{t^{p-1}}-1\right)\end{aligned}$$

- If $p > 1$, then $p-1 > 0$, so as $t \to \infty$, $t^{p-1} \to \infty$ and $1/t^{p-1} \to 0$.

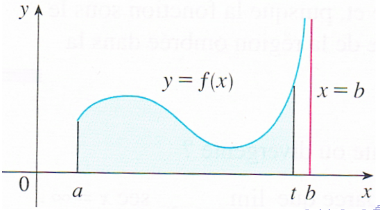
Solution (continued):

- Therefore
$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1},$$
and so the integral converges when $p > 1$.
- But if $p < 1$, then $p - 1 < 0$ and so
$$\frac{1}{t^{p-1}} = t^{1-p} \to \infty \text{ as } t \to \infty,$$
and the integral diverges.

Type 2: Discontinuous Integrands

- Suppose that f is a positive continuous function on a finite interval $[a, b[$ but has a vertical asymptote at b.
- Let S be the unbounded region under the graph of f and above the x-axis between a and b.
- The area of the part of S between a and t (the shaded region in the figure below) is

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx.$$



Definition (Definition of an Improper Integral of Type 2)

(a) *If f is continuous on $[a, b[$ and is discontinuous at b, then*

$$\int_a^b f(x)dx = \lim_{t \to b^-} \left(\int_a^t f(x)dx \right)$$

if this limit exists (as a finite number).

(b) *If f is continuous on $]a, b]$ and is discontinuous at a, then*

$$\int_a^b f(x)dx = \lim_{t \to a^+} \left(\int_t^b f(x)dx \right)$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)dx$ is called **convergent** *if the corresponding limit exists and* **divergent** *if the limit does not exist.*

(c) *If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example

Find

$$\int_{2}^{5} \frac{1}{\sqrt{x-2}}\,dx.$$

Solution:

- We note first that the given integral is improper because $f(x) = \sqrt{x-2}$ has the vertical asymptote $x = 2$.
- Since the infinite discontinuity occurs at the left endpoint of $[2, 5]$, we use part (b) of the previous definition:

$$\begin{aligned}\int_2^5 \frac{dx}{\sqrt{x-2}} &= \lim_{t\to 2^+}\int_t^5 \frac{dx}{\sqrt{x-2}}\\ &= \lim_{t\to 2^+}\left[2\sqrt{x-2}\right]_t^5\\ &= \lim_{t\to 2^+} 2\left(\sqrt{3}-\sqrt{t-2}\right)\\ &= 2\sqrt{3}\end{aligned}$$

- Thus the given improper integral is convergent.

Example

Determine whether

$$\int_0^{\pi/2} \sec x \, dx$$

converges or diverges.

Solution:

- Note that the given integral is improper because $\lim_{x\to(\pi/2)^-} \sec x = \infty$.
- Using part (a) of our previous definition, we have that

$$\begin{aligned}
\int_0^{\pi/2} \sec x\, dx &= \lim_{t\to(\pi/2)^-} \int_0^t \sec x\, dx \\
&= \lim_{t\to(\pi/2)^-} \left[\ln|\sec x + \tan x|\right]_0^t \\
&= \lim_{t\to(\pi/2)^-} \left[\ln(\sec t + \tan t) - \ln 1\right] \\
&= \infty
\end{aligned}$$

  because $\sec t \to \infty$ and $\tan t \to \infty$ as $t \to (\pi/2)^-$.
- Thus the given improper integral is divergent.

Example

Evaluate

$$\int_0^3 \frac{dx}{x-1}$$

if possible.

Solution:

- Observe that the line $x = 1$ is a vertical asymptote of the integrand.
- Since it occurs in the middle of the interval $[0, 3]$, we must use part (c) of the previous definition with $c = 1$:

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

where

$$\begin{aligned}
\int_0^1 \frac{dx}{x-1} &= \lim_{t \to 1^-} \int_0^t \frac{dx}{x-1} \\
&= \lim_{t \to 1^-} \left[\ln|x-1|\right]_0^t \\
&= \lim_{t \to 1^-} \left(\ln|t-1| - \ln|-1|\right) \\
&= -\infty
\end{aligned}$$

because $1 - t \to 0^+$ as $t \to 1^-$.

Solution (continued):

- Thus
$$\int_0^1 \frac{dx}{x-1}$$
is divergent.
- This implies that
$$\int_0^3 \frac{dx}{x-1}$$
is divergent.

WARNING:

- If we had not noticed the asymptote $x = 1$ in the previous example and had instead confused the integral with an ordinary integral, then we might have made the following erroneous calculation:

$$\int_0^3 \frac{dx}{x-1} = [\ln|x-1|]_0^3 = \ln 2 - \ln 1 = \ln 2.$$

- This is wrong because the integral is improper and must be calculated in terms of limits.

A Comparison Test for Improper Integrals

Theorem (Comparison Theorem)

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) *If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.*

(b) *If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.*

Example

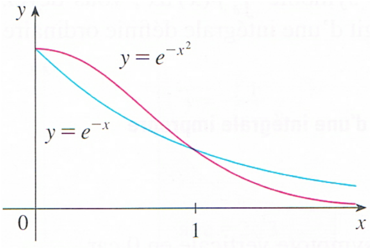
Show that

$$\int_0^\infty e^{-x^2}\,dx$$

is convergent.

Solution:

- We can't evaluate the integral directly because the antiderivative of e^{-x^2} is not an elementary function.
- We write
$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$
and observe that the first integral on the right-hand side is just an ordinary definite integral.
- In the second integral we use the fact that for $x \geq 1$, we have $x^2 \geq x$, so $-x^2 \leq -x$ and therefore $e^{-x^2} \leq e^{-x}$.



Solution (continued):

- The integral of e^{-x} is easier to evaluate:

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx = \lim_{t\to\infty}\left(e^{-1}-e^{-t}\right) = e^{-1}.$$

- Therefore, taking $f(x) = e^{-x}$ and $g(x) = e^{-x^2}$ in the Comparison Theorem, we see that $\int_1^\infty e^{-x^2}\,dx$ is convergent.
- It follows that $\int_0^\infty e^{-x^2}\,dx$ is convergent.

Example

The integral $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ *is divergent by the Comparison Theorem because*

$$\frac{1+e^{-x}}{x} > \frac{1}{x}$$

and $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ *is divergent.*