MATH 1009-E **TEST 2 - solutions** October 2012

PART A: MULTIPLE CHOICE QUESTIONS.

[2 marks each =10 marks]

A1. $\displaystyle\lim_{x\to -1} \frac{2x^3 - x^2 + 7x + 4}{1 - x^5}$ is

[(a) -3] (b) -6. (c) 6. (d) ∞. (e) None of the above.

A2. $\displaystyle\lim_{x\to 2} \frac{4 - x^2}{2 - x}$ is

(a) ∞. [(b) 4] (c) 0. (d) 2. (e) None of the above.

A3. $\displaystyle\lim_{x\to -\infty} \frac{2x^3 + 3x^2 + x - 11}{x^4 + 1}$ is

(a) 2. (b) ∞. (c) $-\infty$. [(d) 0] (e) None of the above.

A4. $\displaystyle\lim_{x\to \infty} \frac{x^4 + x^2 - 9}{x^2 - 3}$ is

(a) -2. (b) $-\infty$. [(c) ∞] (d) 0. (e) None of the above.

A5. The function $g(x) = \dfrac{x^2}{x^2 - 9}$ is discontinuous at

(a) $x = \pm\sqrt{3}$. (b) $x = \sqrt{3}$ [(c) $x = \pm 3$] (d) $x = 3$. (e) None of the above.

PART B: LONG STYLE QUESTIONS.

[7 marks] B1.

Let $f(x) = \begin{cases} 5 & \text{for } x \leq -2, \\ x^2 & \text{for } x > -2. \end{cases}$

[1] (a) Sketch the graph of f.

(b) Find the following limits:

[1] (i) $\lim_{x \to -2+} f(x) = 4$;

[1] (ii) $\lim_{x \to -2-} f(x) = 5$;

[1] (iii) $\lim_{x \to -2} f(x) =$ DNE (Does Not Exist);

[1] (iv) $\lim_{x \to 1} f(x) = 1$;

[1] (v) $\lim_{x \to -3} f(x) = 5$;

[1] (c) Is the function f continuous at $x = -2$? Explain.

Since the left-hand-side limit (i) is not the same as the right-hand-side limit (ii), the limit at -2 does not exist. Thus, the function is discontinuous at $x = -2$. The graph has a gap at $x = -2$. (Note to the markers: a reference to the graph is sufficient to give the full mark.)

[6 marks] B2. Use the Intermediate Value Theorem to show that the equation has at least one solution in the designated interval. Explain why the theorem is applicable.

[3] (a) $x^2 - 6x + 8 = 0$, in $(1, 3)$

[3] (b) $3x = 2^x$, in $(0, 1)$.

Solution: (a) $f(x) = x^2 - 6x + 8$ is a polynomial, therefore, it is continuous on the interval $[1, 3]$. Thus, we can apply the IVT on that interval.

$$f(1) = (1)^2 - 6 \cdot 1 + 8 = 3 > 0; \quad f(3) = 3^2 - 6 \cdot 3 + 8 = -1 < 0.$$

Since the values of the function have opposite signs at the endpoints of the interval, then there is at least one point on the interval, at which this function assumes the value of zero.

(b) In the context of this problem, $f(x) = 3x - 2^x$. As a difference of two continuous function, it is continuous on the interval $[0, 1]$,

$$f(0) = 3 \cdot 0 - 2^0 = 0 - 1 = -1 < 0; \quad f(1) = 3 \cdot 1 - 2^1 = 3 - 2 = 1 > 0.$$

Since the values of the function have opposite signs at the endpoints of the interval, then there is at least one point on the interval, at which this function assumes the value of zero.

[**5 marks**] **B3**. Use implicit differentiation to find $\dfrac{dy}{dx}$ if $x^2 + 4xy + y^3 = 7$.

Solution:

Compute the derivative with respect to x of both sides of the equation :

$$2x + 4 \cdot 1 \cdot y + 4x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0.$$

Solving for $\dfrac{dy}{dx}$ yields

$$\frac{dy}{dx} = -\frac{2x + 4y}{4x + 3y^2}.$$

[**12 marks**] **B4**. Find the derivative of the following functions using the appropriate rules of differentiation:

[**3**] **(a)** $f(x) = (x^5 + 3x^3 + 1)(7x^2 - x - 23)$. (Do not simplify.)

[**3**] **(b)** $g(x) = \dfrac{2x^2}{3x^3 + 5}$. (Simplify the numerator.)

[**3**] **(c)** $h(x) = 3e^{1-2x}$.

[**3**] **(d)** $k(x) = 2(1 - 3x^2)^{1/4}$.

Solution:

(a) $f'(x) = (x^5 + 3x^3 + 1)'(7x^2 - x - 23) + (x^5 + 3x^3 + 1)(7x^2 - x - 23)' =$

$= (5x^4 + 9x^2)(7x^2 - x - 23) + (x^5 + 3x^3 + 1)(14x - 1)$.

(b) $g'(x) = \dfrac{(2x^2)'(3x^3 + 5) - 2x^2(3x^3 + 5)'}{(3x^3 + 5)^2} = \dfrac{4x(3x^3 + 5) - 2x^2 \cdot 9x^2}{(3x^3 + 5)^2} = \dfrac{-6x^4 + 20x}{(3x^3 + 5)^2}$.

(c) $h'(x) = 3 \cdot e^{1-2x} \cdot (1 - 2x)' = 3(-2)e^{1-2x} = -6e^{1-2x}$.

(d) $k'(x) = 2 \cdot \dfrac{1}{4}(1 - 3x^2)^{-3/4} \cdot (1 - 3x^2)' = \dfrac{1}{2}(1 - 3x^2)^{-3/4}(-6x) = -3x(1 - 3x^2)^{-3/4}$.