MATH 105, section 204 — Mon 16 Mar

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Today: sequences

- recurrence relations
- limits of sequences

1

A <u>sequence</u> is an ordered list of numbers

$$a_1, a_2, a_3, \ldots$$

Notation for the whole sequence: $\{a_n\}$

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Example: $a_1 = 21$

$a_2 = 24$

$a_3 = 27$ Can you guess a_4?

2 ways to guess a_4:

(1) $a_4 = 3 + 27$

(2) $a_1 = 3\cdot(1+6)$, $a_2 = 3\cdot(2+6)$, $a_3 = 3\cdot(3+6)$,

so $a_4 = 3\cdot(4+6)$

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A <u>recurrence relation</u> expresses a_{n+1} in terms of a_n. Here, $a_{n+1} = 3 + a_n$.

An <u>explicit formula</u> expresses a_n in terms of n. Here, $a_n = 3(n+6)$.

$\boxed{4}$ Example 2: Find the first four terms of these sequences. Can you find an explicit formula?

sequence 1: $\begin{cases} a_{n+1} = 2a_n + 1 \\ a_1 = 1 \end{cases}$

sequence 2: $\begin{cases} b_{n+1} = 2b_n + 1 \\ b_1 = -1 \end{cases}$

(Try it before looking at the answer.)

$\boxed{5}$ Answer:

$a_1 = 1$
$a_2 = 2 \cdot 1 + 1 = 3$
$a_3 = 2 \cdot 3 + 1 = 7$
$a_4 = 2 \cdot 7 + 1 = 15$

Explicit formula:
$a_n = 2^n - 1$
(by noticing a pattern)

$b_1 = -1$
$b_2 = 2 \cdot (-1) + 1 = -1$
$b_3 = 2 \cdot (-1) + 1 = -1$
$b_4 = 2 \cdot (-1) + 1 = -1$

Explicit formula:
$b_n = -1$ for all n

$\boxed{6}$

<u>Definition.</u> If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases, we write $\lim_{n\to\infty} a_n = L$ and say the sequence <u>converges</u> to L. Otherwise, the sequence <u>diverges</u> (has no limit).

$\boxed{7}$

Examples and nonexamples:

Do these converge to a limit, or do they diverge? Make a guess for each one before looking at the answer.

(1) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

$a_n = \frac{1}{n}$

(2) $1, 0, 1, 0, 1, 0, \ldots$

$a_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

(3) $1, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \frac{1}{7}, 0, \ldots$

$a_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

(4) $1, 2, 3, 4, 5, 6, \ldots$

$a_n = n$

(5) $1, -1, \frac{1}{3}, -1, \frac{1}{5}, -1, \frac{1}{7}, -1, \ldots$

$a_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases}$

(6) $1, \frac{-1}{2}, \frac{1}{4}, \frac{-1}{8}, \frac{1}{16}, \frac{-1}{32}, \ldots$

$a_n = \left(\frac{-1}{2}\right)^n$

$\boxed{8}$ Answers:

(1) converges to 0

(2) diverges

(3) converges to 0

(4) diverges

(5) diverges

(6) converges to 0

$\boxed{9}$ <u>Theorem 8.1 in book</u> Suppose $f(x)$ is a function such that $f(n) = a_n$ for all positive integers n. If $\lim_{x \to \infty} f(x) = L$, then $\lim_{n \to \infty} a_n = L$ also.

Example: $a_n = \dfrac{4n^3}{n^3+1}$

Choose $f(x) = \dfrac{4x^3}{x^3+1}$

Then $\lim_{x \to +\infty} \dfrac{4x^3}{x^3+1} = \lim_{x \to +\infty} \dfrac{4}{1+\frac{1}{x^3}} = \dfrac{4}{1+0} = 4$

divide top and bottom by x^3

So $\lim_{n \to \infty} a_n = 4$.

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<u>Theorem 8.2 in book</u> Suppose the sequences $\{a_n\}$ and $\{b_n\}$ have limits A and B, respectively. Then

(1) $\lim\limits_{n\to\infty}(a_n \pm b_n) = A \pm B$

(2) $\lim\limits_{n\to\infty} c\,a_n = cA$, where c is any real number

(3) $\lim\limits_{n\to\infty} a_n b_n = AB$

(4) $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$, if $B \neq 0$.

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More examples: Do these sequences have limits? What are they?

(a) $\left\{\frac{(-1)^n}{n^2+1}\right\}_{n=1}^{\infty}$

(b) $\{\cos(n\pi)\}_{n=1}^{\infty}$

(c) $\{a_n\}_{n=1}^{\infty}$ where $a_{n+1} = -2a_n$, $a_1 = 1$

Try it first before reading the answer.

$\boxed{12}$

Answers:

(a) $\dfrac{(-1)^n}{n^2+1} = \pm \dfrac{1}{n^2+1}$

$$\lim_{n\to\infty} \frac{1}{n^2+1} = \lim_{x\to\infty} \frac{1}{x^2+1} = 0$$

$$\lim_{n\to\infty} \frac{-1}{n^2+1} = \lim_{x\to\infty} \frac{-1}{x^2+1} = 0$$

Since the two limits agree, the sequence has limit 0.

(b) $\cos(\pi) = -1$

$\cos(2\pi) = 1$

$\cos(3\pi) = -1$

sequence is $-1, 1, -1, 1, -1, 1, \ldots$ and it diverges

(c) $a_1 = 1$

$a_2 = -2$

$a_3 = 4$

$a_4 = -8$

$\vdots$

Terms get further apart, so the sequence diverges.