Carleton University, School of Mathematics and Statistics

MATH 1119 A, fall 2017

Test 2, October 18, 2017

Part 1. True or false questions. Each question carries one mark.

1. Let $u = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ be two vectors. Then $3u - 2v = \begin{bmatrix} 0 \\ 5 \\ -11 \end{bmatrix}$ T F

2. The vector equation
$$x_1 \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$
Has the same solution set as the matrix equation
$$\begin{bmatrix} 0 & 1 & -1 \\ 2 & -1 & 1 \\ -3 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$
T F

3. Let $u = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ be two vectors. Then $-\frac{1}{3}u$ is a linear combination of u and v. T F

4. Let A be a $m \times n$ matrix and u and v two vectors in $\mathbb{R}^n$. Then $A(u + v) = (Au)(Av)$. T F

5. Consider the linear system $Ax = b$, with $A = [a_1\ a_2\ a_3\ \ldots\ldots a_n]$ a $m \times n$ matrix and b a vector in $\mathbb{R}^m$. If the columns of A span $\mathbb{R}^m$, that is $Span\{a_1\ a_2\ a_3\ \ldots\ldots a_n\} = \mathbb{R}^m$, then the system $Ax = b$ has infinitely many solutions. T F

Part 2. Essay questions. Show you work in all the following questions

6. Consider the vectors **[4 marks]**
$$a_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix},\ a_3 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$$

Does the vector b belong to $Span\{a_1\ a_2\ a_3\}$? If yes, then express b as a linear combination of a_1, a_2, a_3. Justify your answers.

The vector b does belong to $Span\{a_1\ a_2\ a_3\}$. Indeed, an echelon matrix of the system $Ax = b$ with $A = [a_1\ a_2\ a_3]$ is

$$\begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Then the system $Ax = b$ has the unique solution $x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. As the vector equation

$$x_1 a_1 + x_2 a_2 + x_3 a_3 = b$$

has the same solution set as the matrix equation $Ax = b$, the vector b can be written as a linear combination of $a_1\ a_2\ a_3$ with the entries of the solution found $x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ as coefficients as follows.

$$b = x_1 a_1 + x_2 a_2 + x_2\ a_3 = a_1 + 2a_2 + 0\ a_3 = a_1 + 2a_2$$

7. Let $A = \begin{bmatrix} 0 & 2 & 2 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ **[6 marks]**

   a. Determine the values of b_1, b_2 and b_3 for which the linear system $Ax = b$ has a solution? Justify your answer.

   b. When of b_1, b_2 and b_3 are assigned values such that the system $Ax = b$ has a solution, write the solution set of this system in parametric vector form.

Answer

a. The echelon matrix of the augmented matrix is

$$\begin{bmatrix} 1 & -1 & -1 & b_3 \\ 0 & 1 & 1 & b_1/2 \\ 0 & 0 & 0 & b_2 + b_3 \end{bmatrix}$$

Therefore, system does not have a solution for all b because for $b_2 + b_3 \neq 0$ the third equation is impossible. The values for which the system has a solution are given by the equation

$b_2 + b_3 = 0$ and b_1 $free$

The reduced echelon matrix of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & b_3 + b_1/2 \\ 0 & 1 & 1 & b_1/2 \\ 0 & 0 & 0 & b_2 + b_3 \end{bmatrix}$$

b. Thus for $b_2 + b_3 = 0$ and b_1 free, the solution set can be written in the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_3 + b_1/2 \\ -x_3 + b_1/2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_3 + b_1/2 \\ b_1/2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

8. Consider the following system **[5 marks]**

$$x_1 + x_2 + x_3 = 4$$
$$x_2 + 2x_3 = 6$$

a. Write the solution set of this system in parametric vector form.

b. Deduce the solution set of the corresponding homogeneous system for question a.

Answer

The augmented matrix of the system is

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 6 \end{bmatrix}$$

Applying the operation $R1 \to R1 - R2$, we get the reduced echelon form

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 6 \end{bmatrix}$$

Thus, the variables x_1 and x_2 are basic variables and x_3 is a free variable.

The solution can be written as

$$x_1 = -2 + x_3$$
$$x_2 = 6 - 2x_3$$

Let $x_3 = s$, then solution can be written as

$$x = \begin{bmatrix} -2 + s \\ 6 - 2s \\ s \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$, where s is any real number.

The parametric vector form of the solution set of the corresponding homogeneous system

$$x_1 + x_2 + x_3 = 0$$
$$x_2 + 2x_3 = 0$$

is given by

$$x = s \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$, where s is any real number.