3.5. Variation of Parameters; 3.6 Cauchy-Euler Differential Equations

3.5. Variation of parameters. We give the theory for second-order DEs. The general case is similar. Consider a second-order linear DE:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

and suppose that the associated homogeneous DE $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ has a general solution

$$y_c = c_1y_1(x) + c_2y_2(x),$$

where y_1 and y_2 are fundamental set of solutions.
In order to find the general solution

$$y = y_c + y_p$$

of the given DE we have to find a particular solution y_p.
Idea. We are looking for y_p in the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

In other words we replace the constants c_1 and c_2 in the general solution of the associated homogeneous DE by functions $u_1(x)$ and $u_2(x)$. We vary these two functions in order to get a particular solution y_p.

Technically, the Method of Variation of Parameters computes the functions $u_1(x)$ and $u_2(x)$. First, place the DE in **standard form**:

$$\begin{aligned}
&a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)\\
&y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = \frac{g(x)}{a_2(x)}\\
&y'' + p(x)y' + q(x)y = f(x), \quad f(x) = g(x)/a_2(x)\,.
\end{aligned}$$

Next, $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$ and we compute:

$$\begin{aligned}
&y_p'' + p(x)y_p' + q(x)y_p = f(x)\\
&[u_1(x)y_1(x) + u_2(x)y_2(x)]'' + p(x)[u_1(x)y_1(x) + u_2(x)y_2(x)]'\\
&+q(x)[u_1(x)y_1(x) + u_2(x)y_2(x)] = f(x)\\
&[u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2' + u_2y_2'']\\
&+p(x)[u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'] + q(x)[u_1(x)y_1(x) + u_2(x)y_2(x)] = f(x)
\end{aligned}$$

By using the fact that y_1 and y_2 are solutions of the associated homogeneous DE the above expression simplifies to:

$$[u_1^{''}y_1 + 2u_1^{'}y_1^{'} + u_2^{''}y_2 + 2u_2^{'}y_2^{'}] + p(x)[u_1^{'}y_1 + u_2^{'}y_2] = f(x).$$

The above equation can be written in the following form:

$$[u_1^{'}y_1 + u_2^{'}y_2]^{'} + [u_1^{'}y_1^{'} + u_2^{'}y_2^{'}] + p(x)[u_1^{'}y_1 + u_2^{'}y_2] = f(x).$$

Then, substituting

$$u_1^{'}y_1 + u_2^{'}y_2 = 0 \quad \Rightarrow \quad [u_1^{'}y_1 + u_2^{'}y_2]^{'} = 0$$

and we obtain the equation:

$$u_1^{'}y_1^{'} + u_2^{'}y_2^{'} = f(x)\,.$$

As a result, $u_1^{'}$ and $u_2^{'}$ are solution of the following linear system of algebraic equations:

$$\begin{aligned} u_1^{'}y_1 + u_2^{'}y_2 &= 0 \\ u_1^{'}y_1^{'} + u_2^{'}y_2^{'} &= f(x) \end{aligned}$$

Hence,

$$u_1^{'} = \frac{W_1}{W}, \quad u_2^{'} = \frac{W_2}{W},$$

where

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}.$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Finally

$$u_1 = \int u_1^{'}(x)dx, \quad u_2 = \int u_2^{'}(x)dx$$

and the particular solution:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Remark. The Method of **Variation of Parameters** is applicable for any linear DE (not only with constant coefficients). However, we have to know a fundamental set of solutions for the associated homogeneous DE. Note that for some problems application of Variation of Parameters is more simple than Undetermined Coefficients but not in general. Sometimes it is wiser to use Undetermined Coefficients rather than Variation of Parameters. After some practice, you will develop the intuition to know when solving a given problem, which method is more appropriate and will simplify the work.

Remark. Note that in order to start with Variation of Parameters first we have to place the DE in standard form in order to obtain the correct function $\mathbf{f(x)}$.

Example. Solve the DE using Variation of Parameters:

$$4y'' + 4y' + y = xe^{-\frac{x}{2}}.$$

Remark. This is Problem 4 from the Section on Undetermined Coefficients. You will see that the Method of Variation of Parameters is more simple for this problem.

Step 1. Solve the associated homogeneous DE:

$$\begin{aligned}
&\Rightarrow 4y'' + 4y' + y = 0 \\
&\Rightarrow 4m^2 + 4m + 1 = 0 \\
&\Rightarrow m_1 = m_2 = \frac{-4 \pm \sqrt{16 - 16}}{8} = -\frac{1}{2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&y_c = c_1 y_1 + c_2 y_2 \\
&y_1 = e^{-\frac{1}{2}x}, \quad y_2 = xe^{-\frac{1}{2}x} \\
&y_c = c_1 e^{-\frac{1}{2}x} + c_2 xe^{-\frac{1}{2}x}.
\end{aligned}$$

Step 2. Prior to use the method of Variation of Parameters, the DE $4y'' + 4y' + y = xe^{-\frac{x}{2}}$ must first be placed in **standard form**:

$$y'' + y' + \frac{1}{4}y = \frac{xe^{\frac{-x}{2}}}{4} \qquad f(x) = \frac{xe^{-\frac{x}{2}}}{4}.$$

Step 3. Compute the Wronskian $W(y_1, y_2)$, and the determinants W_1 and W_2 by using the formulas:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}.$$

Note that $f(x)$ is the function on the right-hand side of the DE once it has been placed in standard form. In our case:

$$f(x) = \frac{xe^{-\frac{x}{2}}}{4}.$$

Then, to obtain y_2', use product rule: $(x)'e^{-\frac{x}{2}} + x(e^{-\frac{1}{2}x})' = e^{-\frac{x}{2}} - \frac{x}{2}e^{-\frac{x}{2}}$.

$$W = \begin{vmatrix} e^{-\frac{1}{2}x} & xe^{-\frac{1}{2}x} \\ -\frac{1}{2}e^{-\frac{1}{2}x} & e^{-\frac{x}{2}} - \frac{1}{2}xe^{-\frac{x}{2}}. \end{vmatrix}$$

Computing W:

$$\Rightarrow W = e^{-x} - \frac{1}{2}xe^{-x} + \frac{1}{2}xe^{-x} = e^{-x}.$$

$$W_1 = \begin{vmatrix} 0 & xe^{\frac{-x}{2}} \\ \frac{1}{4}xe^{-\frac{x}{2}} & e^{-\frac{x}{2}} - \frac{1}{2}xe^{-\frac{x}{2}} \end{vmatrix}.$$

$$\Rightarrow W_1 = \left(-\frac{1}{4}xe^{-\frac{x}{2}}\right)(xe^{-\frac{x}{2}} = -\frac{1}{4}x^2e^{-x}.$$

$$W_2 = \begin{vmatrix} e^{-\frac{x}{2}} & 0 \\ -\frac{1}{2}e^{-\frac{x}{2}} & \frac{1}{4}xe^{-\frac{x}{2}} \end{vmatrix}.$$

$$\Rightarrow W_2 = \frac{1}{4}xe^{-x}.$$

Step 4. Compute u_1 and u_2. Here we have to integrate:

$$u_1 = \int \frac{W_1}{W}dx \Rightarrow \int \frac{-x^2}{4}e^{-x}\left(\frac{1}{e^{-x}}\right)dx = \int \frac{-x^2}{4}dx = -\frac{1}{12}x^3.$$

$$u_2 = \int \frac{W_2}{W}dx \Rightarrow \int \frac{1}{4}xe^{-x}\left(\frac{1}{e^{-x}}\right)dx = \int \frac{1}{4}x\,dx = \frac{1}{8}x^2.$$

Step 5. Form the particular solution y_p:

$$y_p = u_1y_1 + u_2y_2.$$

Therefore,

$$y_p = \left(\frac{x^2}{8}\right)\left(xe^{-\frac{x}{2}}\right) + \left(-\frac{x^3}{12}\right)\left(e^{-\frac{x}{2}}\right)$$

$$y_p = \frac{x^3}{8}e^{-\frac{x}{2}} - \frac{x^3}{12}e^{-\frac{x}{2}}$$

$$\Rightarrow \quad y_p = \frac{x^3}{24}e^{-\frac{x}{2}}.$$

Step 6. Construct the general solution of the given DE by using y_c and y_p:

$$y = y_c + y_p$$
$$y = c_1 y_1 + c_2 y_2 + y_p$$
$$y_1 = e^{-\frac{1}{2}x}, \quad y_2 = xe^{-\frac{1}{2}x}, \quad y_p = \frac{x^3}{24}e^{-\frac{x}{2}}$$
$$y = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x} + \frac{x^3}{24}e^{-\frac{x}{2}}.$$

Problem 1. Find the general solution of the DE $y'' - y = x^2$ by using variation of parameters.
Solution. First, we solve the homogenous:

$$y'' - y = 0.$$
$$m^2 - 1 = 0 \Rightarrow m_{1,2} = \pm 1.$$
$$y_c = c_1 e^{-x} + c_2 e^{x}.$$

Next, we replace the constants c_1 and c_2 with functions $u_1(x)$ and $u_2(x)$ and we look for a y_p in the form: $y_p = u_1 e^{-x} + u_2 e^{x}$. In other words, we vary the parameters c_1 and c_2, replacing them by functions u_1 and u_2, as shown in the Example problem above.

Note that the DE is given in standard form and we proceed by finding W, W_1, W_2, u_1, and u_2:

$$W = \begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix} \quad \Rightarrow \quad W = 2.$$

$$W_1 = \begin{vmatrix} 0 & e^{x} \\ x^2 & e^{x} \end{vmatrix} \quad \Rightarrow \quad W_1 = -x^2 e^{x}.$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & x^2 \end{vmatrix} \quad \Rightarrow \quad W_2 = x^2 e^{-x}.$$

Then,

$$u_1 = \int \frac{-x^2e^x}{2} = \frac{-1}{2}\int x^2e^x\,dx \text{ (integration by parts)}$$
$$\Rightarrow \frac{-1}{2}x^2e^x + \frac{1}{2}\int e^x 2x\,dx = \frac{-1}{2}x^2e^x + \int e^x x\,dx = \frac{-1}{2}x^2e^x + xe^x - e^x.$$
$$\Rightarrow u_1 = \left(-\frac{x^2}{2} + x - 1\right)e^x.$$
$$u_2 = \int \frac{x^2e^{-x}}{2} = \frac{1}{2}\int x^2e^{-x}\,dx = \frac{-1}{2}x^2e^{-x} + \frac{1}{2}\int e^{-x}2x\,dx$$
$$\Rightarrow -\frac{1}{2}x^2e^{-x} + xe^{-x} + \int e^{-x}\,dx = -\frac{1}{2}x^2e^{-x} + xe^{-x} - e^{-x}$$
$$\Rightarrow u_2 = \left(\frac{-x^2}{2} - x - 1\right)e^{-x}$$

Since $y_p = u_1y_1 + u_2y_2$,

$$y_p = \left(\frac{-x^2}{2} + x - 1\right)e^xe^{-x} + \left(\frac{-x^2}{2} - x - 1\right)e^xe^{-x} \Rightarrow -x^2 - 2.$$

Thus, the general solution is:

$$y = y_c + y_p$$
$$y = c_1y_1 + c_2y_2 + y_p$$
$$y = c_1e^{-x} + c_2e^x - x^2 - 2.$$

Problem 2. Solve the following differential equation

$$y'' + 2y' + y = xe^{-x}$$

by variation of parameters.

Solution. First, solve the associated homogeneous:

$$m^2 + 2m + 1 = 0 \quad \Rightarrow \quad (m+1)^2 = 0.$$
$$\Rightarrow \quad m_1 = m_2 = -1 \quad \Rightarrow \quad y_c = c_1e^{-x} + c_2xe^{-x}\,.$$

Note that the DE is given in standard form. So, we proceed to find W, W_1, W_2, $u_1^{'}$, $u_2^{'}$ and u_1, u_2:

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} \quad \Rightarrow \quad W = e^{-2x}.$$

$$W_1 = \begin{vmatrix} 0 & xe^{-x} \\ xe^{-x} & e^{-x} - xe^{-x} \end{vmatrix} \quad \Rightarrow \quad W_1 = -x^2 e^{-2x}.$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & xe^{-x} \end{vmatrix} \quad \Rightarrow \quad W_2 = xe^{-2x}.$$

Then,

$$u_1 = \int \frac{W_1}{W} dx = \int \frac{-x^2 e^{-2x}}{e^{-2x}} dx = \int -x^2 dx = -\frac{x^3}{3}.$$

$$u_2 = \int \frac{W_2}{W} dx = \int \frac{xe^{-2x}}{e^{-2x}} dx = \int x dx = \frac{x^2}{2}.$$

Since $y_p = u_1 y_1 + u_2 y_2$,

$$y_p = -\frac{x^3}{3} e^{-x} + \frac{x^2}{2}(xe^{-x})$$

$$y_p = -\frac{x^3}{3} e^{-x} + \frac{x^3}{2} e^{-x}$$

$$y_p = \frac{x^3}{6} e^{-x}.$$

The general solution of the given DE is:

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{x} + \frac{x^3}{6} e^{-x}.$$

Problem 3. Solve by using variation of parameters:

$$y'' + y = x.$$

Solution. First, find the general solution of the associated homogeneous DE:

$$m^2 + 1 = 0$$

$$\Rightarrow m \pm i$$

$$\Rightarrow y_c = c_1 e^{-ix} + c_2 e^{ix} = c_1 \cos(x) + c_2 \sin(x).$$

The DE is in standard form. So, find W, W_1, W_2 and $u_1,\ u_2$:

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin x & \cos x \end{vmatrix} \quad \Rightarrow \quad W = \cos^2(x) + \sin^2(x) = 1.$$

$$W_1 = \begin{vmatrix} 0 & \sin(x) \\ x & \cos(x) \end{vmatrix} \quad \Rightarrow \quad W_1 = -x\sin x.$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin(x) & \cos(x) \end{vmatrix} \quad \Rightarrow \quad W_2 = x\cos(x).$$

Then,

$$u_1 = -\int x\sin(x)\,dx = x\cos(x) - \int \cos(x)\,dx = x\cos(x) - \sin(x).$$

$$u_2 = \int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx = x\sin(x) + \cos(x).$$

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 = (x\cos(x) - \sin(x))\cos(x) + (x\sin(x) + \cos(x))\sin(x) \\ y_p &= x(\cos^2(x) + \sin^2(x)) - \sin(x)\cos(x) + \cos(x)\sin(x) \\ y_p &= x \end{aligned}$$

The general solution $y = y_c + y_p$:

$$y = c_1\cos(x) + c_2\sin(x) + x.$$

Problem 4. Solve the following IVP by variation of parameters:

$$\begin{aligned} &y'' + 4y' + 4y = (3+x)e^{-2x} \\ &y(0) = 2, \quad y'(0) = 5\,. \end{aligned}$$

Solution. First, solve the associated homogenous DE:

$$m^2 + 4m + 4 = 0 \Rightarrow (m+2)^2 = 0 \Rightarrow m_{1,2} = -2 \Rightarrow y_c = c_1e^{-2x} + c_2xe^{-2x}.$$

Since the DE is in standard form, proceed to find W, W_1, W_2 and u_1, u_2:

$$W = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-x} & (1-2x)e^{-2x} \end{vmatrix} \quad \Rightarrow \quad W = e^{-4x} - 2xe^{-4x} + 2xe^{-4x} = e^{-4x}.$$

$$W_1 = \begin{vmatrix} 0 & xe^{-2x} \\ (3+x)e^{-2x} & (1-2x)e^{-2x} \end{vmatrix} \quad \Rightarrow \quad W_1 = -x(3+x)e^{-4x}.$$

$$W_2 = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & (3+x)e^{-2x} \end{vmatrix} \quad \Rightarrow \quad W_2 = (3+x)e^{-4x}\,.$$

Then,

$$
\begin{aligned}
u_1 &= \int \frac{-x(3+x)e^{-4x}}{e^{-4x}}dx = \int -(3x+x^2)\,dx = -\frac{3}{2}x^2 - \frac{x^3}{3}.\\
u_2 &= \int \frac{(3+x)e^{-4x}}{e^{-4x}}dx = \int (x+3)\,dx = \frac{x^2}{2} + 3x.\\
\Rightarrow y_p &= \left(-\frac{3}{2}x^2 - \frac{x^3}{3}\right)e^{-2x} + \left(\frac{x^2}{2} + 3x\right)xe^{-2x}\\
\Rightarrow y_p &= \left(3x^2 - \frac{3}{2}x^2\right)e^{-2x} + \left(\frac{x^3}{2} - \frac{x^3}{3}\right)e^{-2x}\\
\Rightarrow y_p &= \frac{3}{2}x^2e^{-2x} + \frac{x^3}{6}e^{-2x}.
\end{aligned}
$$

The general solution of the given equation is

$$
\begin{aligned}
&y = y_c + y_p\\
&y = c_1e^{-2x} + c_2xe^{-2x} + \left(\frac{3}{2}x^2 + \frac{x^3}{6}\right)e^{-2x}
\end{aligned}
$$

Now, apply the initial-value conditions of the initial-value problem:

$$y(0) = c_1 + (c_2)(0) + 0 + 0 = 2 \quad \Rightarrow c_1 = 2.$$

$$y'(0) = -2c_1 + c_2 = 5 \quad \Rightarrow c_2 = 5 + 2c_1 = 9.$$

Thus, the unique solution of the given IVP is:

$$y = 2e^{-2x} + 9xe^{-2x} + \frac{x^3}{6}e^{-2x} + \frac{3x^2}{2}e^{-2x}.$$

Problem 5. Using variation of parameters, find the general solution of

$$y'' - 3y^{'} + 2y = e^{2x} + xe^{x}.$$

Solution. First, solve the associated homogeneous DE:

$$m^2 - 3m + 2 = 0 \Rightarrow m_{1,2} = \frac{3 \pm \sqrt{9-8}}{2} = 1, 2 \Rightarrow y_c = c_1e^{x} + c_2e^{2x}.$$

We look for y_p in the form: $y_p = u_1 e^x + u_2 e^{2x}$. **Since the DE is in standard form**, we proceed to find W, W_1, W_2 and u_1, u_2:

$$W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} \quad \Rightarrow \quad W = e^{3x}\,.$$

$$W_1 = \begin{vmatrix} 0 & e^{2x} \\ e^{2x} + xe^x & 2e^{2x} \end{vmatrix} \quad \Rightarrow \quad W_1 = -e^{4x} - xe^{3x}\,.$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & e^{2x} + xe^x \end{vmatrix} \quad \Rightarrow \quad W_2 = e^{3x} + xe^{2x}\,.$$

Then,

$$u_1 = \int \frac{-e^{4x} - xe^{3x}}{e^{3x}}\,dx = \int -e^x - x\,dx = -e^x - \frac{x^2}{2}.$$

$$u_2 = \int \frac{e^{3x} + xe^{2x}}{e^{3x}}\,dx = \int 1 + xe^{-x}\,dx = x - xe^{-x} - e^{-x}.$$

$$y_p = u_1 e^x + u_2 e^{2x} = (-e^x - \frac{x^2}{2})e^x + (x - xe^{-x} - e^{-x})e^{2x}.$$

$$\Rightarrow -e^{2x} - \frac{x^2}{2}e^x + xe^{2x} - xe^x - e^x = -\frac{x^2}{2}e^x - xe^x + xe^{2x} + (-e^{2x} - e^x).$$

Note that $(-e^{2x} - e^x)$ is a solution of the homogeneous DE and add this term to y_c. Thus, the general solution is:

$$y = c_1 e^{-2x} + c_2 xe^{-2x} - \frac{x^2}{2}e^x - xe^x + xe^{2x} + (-e^{2x} - e^x)$$

$$\Rightarrow -\frac{x^2}{2}e^x - xe^x + xe^{2x} + (c_1 - 1)e^x + (c_2 - 1)e^{2x}.$$

Since $(c_1 - 1)$ is an arbitrary constant, we denote it by c_1, and analogously, $(c_2 - 1)$ is denoted by c_2. Finally, the general solution:

$$y = y_c + y_p$$

$$y = c_1 e^x + c_2 e^{2x} - \frac{x^2}{2}e^x - xe^x + xe^{2x}$$

3.6. Cauchy-Euler differential equations. We shall consider second-order Cauchy-Euler DE. The study of higher order DEs is similar. Second-order Cauchy-Euler DE has the form:

$$[a_2x^2]y^{''} + [a_1x]y^{'} + a_0y = g(x), \quad x > 0$$

where a_2, a_1, a_0 are constants (numbers). Note that the coefficients are non-constant, i.e., they are function but of special form.

Solution of homogeneous, second-order Cauchy-Euler DE:

$$a_2x^2y^{''} + a_1xy^{'} + a_0y = 0, \quad x > 0.$$

Looking for a solution in a form $y = x^m$ and computing $y^{'} = mx^{m-1}$, $y^{''} = m(m-1)x^{m-2}$ we obtain:

$$\begin{aligned}
&a_2x^2y^{''} + a_1xy^{'} + a_0y = 0\\
&a_2x^2(m(m-1)x^{m-2}) + a_2(mx^{m-1}) + a_0x^m\\
&x^m\left[a_2m(m-1) + a_1m + a_0\right] = 0, \quad x > 0\\
&a_2m(m-1) + a_1m + a_0 = 0\\
&a_2m^2 + (a_1 - a_2)m + a_0 = 0
\end{aligned}$$

that is a quadratic equation called **characteristic equation of a homogeneous second-order Cauchy-Euler DE**. According to the solutions m_1 and m_2 of the characteristic equation we have the following 3 cases:

(a) Distinct real solutions $m_1 \neq m_2$. Then the general solution is:

$$y = c_1x^{m_1} + c_2x^{m_2}.$$

(b) Repeated real solutions $m_1 = m_2$. Then the general solution is:

$$y = c_1x^{m_1} + c_2x^{m_1}\ln(x).$$

(c) Complex (conjugate) solutions $m_1 = \alpha - i\beta$, $m_2 = \alpha + i\beta$. Then **the general solution in complex form** is:

$$\begin{aligned}
y &= c_1x^{m_1} + c_2x^{m_2}\\
y &= c_1x^{\alpha-i\beta} + c_2x^{\alpha+i\beta}\\
y &= c_1x^{\alpha}e^{-i\beta\ln(x)} + c_2x^{\alpha}e^{i\beta\ln(x)}
\end{aligned}$$

and from here **the real form of the general solution** is:

$$\mathbf{y} = \mathbf{x}^{\alpha}\left[\mathbf{c_1}\cos(\beta\ln(\mathbf{x})) + \mathbf{c_2}\sin(\beta\ln(\mathbf{x}))\right].$$

General solution of a non-homogeneous, second-order, Cauchy-Euler DE:

$$a_2x^2y^{''} + a_1xy^{'} + a_0y = g(x), \quad x > 0.$$

(1) Find the general solution y_c of the associated homogeneous DE.
(2) By using **the method variation of parameters** find a particular solution y_p of the given non-homogeneous DE. **Note that the method of undetermined coefficients is not applicable to find y_p for Cauchy-Euler DE because Cauchy-Euler DE is not with constant coefficients!**
(3) Form the general solution of the given non-homogeneous Cauchy-Euler DE:

$$y = y_c + y_p.$$

Example 1. Solve the homogeneous Cauchy-Euler DE:

$$x^2y'' - 4xy' + 4y = 0.$$

Step 1. Substitute $y = x^m$ into the DE $x^2y'' - 4xy' + 4y = 0$.

$$y = x^m, \quad y' = mx^{m-1}, \quad y'' = m(m-1)x^{m-2}.$$

Plugging these values into the DE:

$$x^2(m^2 - m)x^{m-2} - 4x(mx^{m-1}) + 4x^m.$$
$$\Rightarrow (m^2 - m)x^m - 4mx^m + 4x^m = 0.$$
$$\Rightarrow m^2 - m - 4m + 4 = 0.$$
$$\Rightarrow (m-4)(m-1) = 0.$$
$$\Rightarrow m_1 = 1 \text{ and } m_2 = 4.$$

Step 2. We have two real distinct solutions. Hence, the general solution has the form $y = c_1x^{m_1} + c_2x^{m_2}$. Thus, the general solution of the given homogeneous Cauchy-Euler DE is:

$$y = c_1x + c_2x^4.$$

Example 2. Solve the non-homogeneous Cauchy-Euler DE:

$$x^2y'' + xy' - y = \ln(x).$$

Step 1. Solve the associated homogeneous DE:

$$x^2y'' + xy' - y = 0.$$

Since the expression contains variable coefficients, must solve for y_c by Cauchy Euler. Thus, substitute $y = x^m, y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$ into the DE:

$$\begin{aligned} &m^2 - m + m - 1 = 0 \\ &\Rightarrow m^2 - 1 = 0 \\ &\Rightarrow m^2 = 1 \\ &\Rightarrow m = \pm 1 \end{aligned}$$

Thus, $y_c = c_1x + c_2x^{-1}$.
Step 2. Find a particular solution y_p of the given non-homogeneous DE. **Note that since the coefficients of the DE are variable (non-constant), we cannot use the method of undetermined coefficients. If the coefficients of the DE are variable, we apply the method variation of parameters.** Thus, put the DE in **standard form**:

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = \frac{\ln(x)}{x^2}.$$

$$W = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} \quad \Rightarrow \quad W = -x^{-1} - x^{-1} = -2x^{-1} = \frac{-2}{x}.$$

$$W_1 = \begin{vmatrix} 0 & x^{-1} \\ \frac{\ln(x)}{x^2} & -x^{-2} \end{vmatrix} \quad \Rightarrow \quad W_1 = \left(\frac{-1}{x}\right)\left(\frac{\ln(x)}{x^2}\right) = \frac{-\ln(x)}{x^3}.$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & \frac{\ln(x)}{x^2} \end{vmatrix} \quad \Rightarrow \quad W_2 = \left(\frac{\ln(x)}{x^2}\right)(x) = \frac{\ln(x)}{x}.$$

Then, find u_1 and u_2, solving both integrals using integration by parts:

$$u_1 = \int \frac{-\ln(x)}{x^3}\left(\frac{x}{-2}\right)dx = \frac{1}{2}\int \frac{1}{x^2}\ln(x)dx = \frac{-\ln(x)}{2x} - \frac{1}{2x}.$$

$$u_2 = \int \frac{\ln(x)}{x}\left(\frac{x}{-2}\right)dx = \frac{-1}{2}\int \ln(x)dx = -\frac{1}{2}(x\ln(x) - x).$$

Find $y_p = u_1y_1 + u_2y_2$:

$$\left(\frac{-\ln(x) - 1}{2x}\right)(x) + \left(\frac{-x}{2}\ln(x) + \frac{x}{2}\right)\left(\frac{1}{x}\right)$$

$$\Rightarrow y_p = \frac{-\ln(x) - 1}{2} + \left(\frac{-\ln(x)}{2} + \frac{1}{2}\right)$$
$$\Rightarrow y_p = \frac{-\ln(x) - 1 - \ln(x) + 1}{2} \Rightarrow \frac{-2\ln(x)}{2} \Rightarrow -\ln(x)$$

Step 3. Construct **the general solution** in the form $y = y_c + y_p$:

$$y = c_1 x + c_2 x^{-1} - \ln(x).$$

Problem 1. Solve the following homogeneous Cauchy-Euler DE:

$$x^2 y'' + xy' + y = 0, \quad x > 0.$$

Solution. Looking for a solution in a form $y = x^m$ we obtain:

$$\begin{aligned} &x^2 y'' + xy' + y = 0 \\ &x^2 m(m-1)x^{m-2} + xmx^{m-1} + x^m = 0 \\ &x^m [m(m-1) + m + 1] = 0 \\ &m^2 + 1 = 0 \quad \Rightarrow \quad m_{1,2} = \pm i\,. \end{aligned}$$

We have the case of two complex conjugate solutions $m_1 = -i$, $m_2 = i$. Complex form of the general solution:

$$y = c_1 x^{-i} + c_2 x^i.$$

Real form of the general solution ($\alpha = 0$, $\beta = 1$):

$$\begin{aligned} &y = x^\alpha [c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x))] \\ &y = x^0 [c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))] \\ &y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)) \end{aligned}$$

Problem 2. Solve the following homogeneous Cauchy-Euler DE:

$$x^2 y'' + 3xy' + 2y = 0, \quad x > 0.$$

Solution. Looking for a solution in a form $y = x^m$ we obtain:

$$\begin{aligned} &x^2 y'' + 3xy' + 2y = 0 \\ &x^2 m(m-1)x^{m-2} + 3xmx^{m-1} + 2x^m = 0 \\ &x^m [m(m-1) + 3m + 2] = 0 \\ &m^2 + 2m + 2 = 0 \quad \Rightarrow \quad m_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} \\ &m_1 = -1 - i, \quad m_2 = -1 + i \end{aligned}$$

We have the case of two complex conjugate solutions $m_1 = -i$, $m_2 = i$. Complex form of the general solution:

$$y = c_1 x^{-1-i} + c_2 x^{-1+i}.$$

Real form of the general solution ($\alpha = -1$, $\beta = 1$):

$$\begin{aligned} y &= x^{\alpha} \left[c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x))\right] \\ y &= x^{-1} \left[c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))\right] \end{aligned}$$

Problem 3. Solve the following homogeneous Cauchy-Euler DE:

$$x^2 y^{''} + 3xy^{'} + y = 0, \quad x > 0.$$

Solution. Looking for a solution in a form $y = x^m$ we obtain:

$$\begin{aligned} & x^2 y^{''} + 3xy^{'} + y = 0 \\ & x^2 m(m-1)x^{m-2} + 3xmx^{m-1} + x^m = 0 \\ & x^m \left[m(m-1) + 3m + 1\right] = 0 \\ & m^2 + 2m + 1 = 0 \quad \Rightarrow \quad m_{1,2} = \frac{-2 \pm \sqrt{4-4}}{2} \\ & m_1 = m_2 = -1\,. \end{aligned}$$

We have the case of repeated real solutions $m_1 = m_2 = -1$. the general solution:

$$\begin{aligned} y &= c_1 x^{m_1} + c_2 x^{m_1} \ln(x). \\ y &= c_1 x^{-1} + c_2 x^{-1} \ln(x) \end{aligned}$$

Problem 4. Solve the following non-homogeneous Cauchy Euler DE:

$$x^2 y'' + 4xy' + 2y = 4\ln(x), \quad 0 < x < \infty.$$

Solution. First, solve the associated homogeneous DE, by substituting $y = x^m$.

$$\begin{aligned} & x^2 y'' + 4xy' + 2y = 0 \\ & \Rightarrow x^2 m(m-1)x^{m-2} + 4xmx^{m-1} + 2x^m = 0 \\ & \Rightarrow x^m(m(m-1) + 4m + 2) = 0 \\ & \Rightarrow m^2 + 3m + 2 = 0 \\ & \Rightarrow m_{1,2} = \frac{-3 + \sqrt{9-8}}{2} \Rightarrow m_1 = -2, m_2 = -1. \end{aligned}$$

Thus,

$$y_c = c_1 x^{-2} + c_2 x^{-1}.$$

Second, we need to find a particular solution y_p. **Note that here the method of undetermined coefficients is not applicable (the DE is not with constant coefficients!). Thus, we apply the method of variation of parameters. To do this, make sure the DE is in standard form:**

$$y'' + \frac{4}{x}y' + \frac{2}{x^2}y = \frac{4\ln(x)}{x^2} \quad \Leftrightarrow \quad y'' + \frac{4}{x}y' + \frac{2}{x^2}y = 4x^{-2}\ln(x).$$

From y_c, we denote $y_1 = x^{-2}$ and $y_2 = x^{-1}$.
Next, compute W, W_1 and W_2:

$$W = \begin{vmatrix} x^{-2} & x^{-1} \\ -2x^{-3} & -x^{-2} \end{vmatrix} \quad \Rightarrow \quad W = x^{-4}.$$

$$W_1 = \begin{vmatrix} 0 & x^{-1} \\ 4x^{-2}\ln(x) & -x^{-2} \end{vmatrix} \quad \Rightarrow \quad W_1 = -4x^{-3}\ln(x).$$

$$W_2 = \begin{vmatrix} x^{-2} & 0 \\ -2x^{-3} & 4x^{-2}\ln(x) \end{vmatrix} \quad \Rightarrow \quad W_2 = 4x^{-4}\ln(x).$$

Next, find u_1 and u_2:

$$u_1 = \int \frac{-4x^{-3}\ln(x)}{x^{-4}}dx = -4\int x\ln(x)dx$$

Using integration by parts:

$$u_1 = -4\int x\ln(x)dx = -4\int \ln(x)d\frac{x^2}{2} = -4\ln(x)\frac{x^2}{2} + 4\int \frac{x^2}{2}d\ln(x)$$

$$\Rightarrow -2x^2\ln(x) + 2\int x^2\left(\frac{1}{x}\right)dx = -2x^2\ln(x) + x^2$$

$$u_2 = \int \frac{4x^{-4}\ln(x)}{x^{-4}}dx = \int 4\ln(x)\,dx \text{ (integration by parts) } = 4x\ln(x) - 4x.$$

Next, find y_p, knowing that $y_p = u_1 y_1 + u_2 y_2$:

$$y_p = (-2x^2\ln(x) + x^2)x^{-2} + (4x\ln(x) - 4(x))x^{-1}$$

$$\Rightarrow -2\ln(x) + 1 + 4\ln(x) - 4 = 2\ln(x) - 3.$$

Finally, we construct the general solution $y = y_c + y_p$:

$$y = c_1 x^{-2} + c_2 x^{-1} + 2\ln(x) - 3.$$

Problem 5. Find the general solution of the non-homogeneous Cauchy-Euler DE:

$$2x^2 y'' + 5xy' + y = x^2 - x, \quad 0 < x < \infty$$

Solution. First, solve the associated homogeneous DE, substituting $y = x^m$:

$$\begin{aligned}
&2x^2 y'' + 5xy' + y = 0. \\
&\Rightarrow 2x^2 m(m-1)x^{m-2} + 5xmx^{m-1} + x^m = 0
\end{aligned}$$

The x^m cancels out because $x > 0$:

$$\begin{aligned}
&\Rightarrow 2m^2 + 3m + 1 = 0 \\
&\Rightarrow m_{1,2} = \frac{-3 \pm \sqrt{9-8}}{4} \\
&\Rightarrow m_1 = -1, m_2 = -\frac{1}{2}.
\end{aligned}$$

So, $y_c = c_1 x^{-1} + c_2 x^{\frac{-1}{2}}$. Denote $y_1 = x^{-1}$ and $y_2 = x^{-\frac{1}{2}}$.
Next, use variation of parameters to find a particular solution y_p. Thus, put the DE in standard form:

$$\begin{aligned}
y'' + \frac{5}{2x}y' + \frac{1}{2x^2}y &= \frac{x^2 - x}{2x^2} = \frac{1}{2} - \frac{1}{2x}. \\
\Rightarrow y'' + \frac{5}{2x}y' + \frac{1}{2x^2}y &= \frac{1}{2} - \frac{1}{2}x^{-1}.
\end{aligned}$$

Compute W, W_1 and W_2:

$$W = \begin{vmatrix} x^{-1} & x^{-\frac{1}{2}} \\ -x^{-2} & -\frac{1}{2}x^{-\frac{3}{2}} \end{vmatrix} \quad \Rightarrow \quad W = -\frac{1}{2}x^{-\frac{5}{2}} + x^{-\frac{5}{2}} = \frac{1}{2}x^{-\frac{5}{2}}.$$

$$W_1 = \begin{vmatrix} 0 & x^{-\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2}x^{-1} & -\frac{-1}{2}x^{-\frac{3}{2}s} \end{vmatrix} \quad \Rightarrow \quad W_1 = -\left(\frac{1}{2} - \frac{1}{2}x^{-1}\right)x^{-\frac{1}{2}}.$$

$$W_2 = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & \frac{1}{2} - \frac{1}{2}x^{-1} \end{vmatrix} \quad \Rightarrow \quad W_2 = x^{-1}\left(\frac{1}{2} - \frac{1}{2}x^{-1}\right).$$

Find u_1 and u_2:

$$u_1 = \int \frac{-\left(\frac{1}{2} - \frac{1}{2}x^{-1}\right) x^{-\frac{1}{2}}}{\frac{1}{2}x^{-\frac{5}{2}}} dx$$

$$\Rightarrow \int (x^{-1} - 1)x^2 \, dx = \int (x - x^2) \, dx = \frac{x^2}{2} - \frac{x^3}{3}.$$

$$u_2 = \int \frac{x^{-1}\left(\frac{1}{2} - \frac{1}{2}x^{-1}\right)}{\frac{1}{2}x^{-\frac{5}{2}}} \, dx.$$

$$\Rightarrow \int x^{\frac{3}{2}}(1 - x^{-1}) = \int \left(x^{\frac{3}{2}} - x^{\frac{1}{2}}\right) \, dx = \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}}.$$

Using the formula $y_p = u_1y_1 + u_2y_2$ we find a particular solution y_p:

$$y_p = \left(\frac{x^2}{2} - \frac{x^3}{3}\right) x^{-1} + \left(\frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right) x^{-\frac{1}{2}}$$

$$\Rightarrow \frac{x}{2} - \frac{x^2}{3} + \frac{2}{5}x^2 - \frac{2}{3}x = \frac{x^2}{15} - \frac{x}{6}.$$

The general solution has the form

$$y = y_c + y_p = c_1y_1 + c_2y_2 + y_p.$$

Thus, the general solution is:

$$y = c_1x^{-1} + c_2x^{-\frac{1}{2}} + \frac{x^2}{15} - \frac{x}{6}.$$