MAT3320X Assignment 1

Total: 20 marks. Due date: May 28th, 11pm.

(a) (8 marks) Solve the following equations by power series method, your final answers should be simple functions:
(i) $y' = -2xy$. (1)
(ii) $(1-x^2)y' = 2xy$. (2).

Solution: (i) Suppose

$$y(x) = \sum_{m=0}^{\infty} a_m x^m.$$

Then

$$y'(x) = \sum_{m=1}^{\infty} m a_m x^{m-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m.$$

Substituting these into (1) we have

$$\begin{aligned} \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m &= -2x \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} (-2a_m) x^{m+1} \\ &= \sum_{m=1}^{\infty} (-2a_{m-1}) x^m \end{aligned}$$

Collecting like powers of x we get

$$a_1 + \sum_{m=1}^{\infty} \left[(m+1)a_{m+1} + 2a_{m-1}\right] x^m = 0.$$

By the property of VAC,

$$a_1 = 0, \quad (m+1)a_{m+1} + 2a_{m-1} = 0, \quad m = 1, 2, ...$$

This implies that

$$a_1 = a_3 = a_5 = \cdots = 0$$

and

$$a_2 = -a_0, \quad a_4 = -\frac{2}{4} a_2 = \frac{2 \cdot 2}{4 \cdot 2} a_0 = \frac{1}{2!} a_0.$$

In general,

$$a_{2m} = (-1)^m \frac{1}{m!} a_0.$$

Hence

$$\begin{aligned} y &= a_0 - a_0x^2 + a_0\frac{1}{2!}x^4 - + \cdots = a_0 + \sum_{m=1}^{\infty}(-1)^m\frac{1}{m!}a_0x^{2m} \\ &= a_0 + \sum_{m=1}^{\infty}\frac{1}{m!}a_0\left(-x^2\right)^m = a_0\left[1 + \sum_{m=1}^{\infty}\frac{1}{m!}\left(-x^2\right)^m\right] \\ &= a_0e^{-x^2}. \end{aligned}$$

(ii) Suppose

$$y(x) = \sum_{m=0}^{\infty} a_mx^m.$$

Then

$$y'(x) = \sum_{m=1}^{\infty} ma_mx^{m-1} = \sum_{m=0}^{\infty}(m+1)a_{m+1}x^m.$$

Substituting these into (2) we have

$$\begin{aligned} (1-x^2)\sum_{m=0}^{\infty}(m+1)a_{m+1}x^m &= 2x\sum_{m=0}^{\infty}a_mx^m \\ &= \sum_{m=0}^{\infty}(2a_m)x^{m+1} \\ &= \sum_{m=1}^{\infty}(2a_{m-1})x^m \end{aligned}$$

Note that left hand side can be written as:

$$\begin{aligned} \text{LS} &= \sum_{m=0}^{\infty}(m+1)a_{m+1}x^m - \sum_{m=0}^{\infty}(m+1)a_{m+1}x^{m+2} \\ &= \sum_{m=0}^{\infty}(m+1)a_{m+1}x^m - \sum_{m=2}^{\infty}(m-1)a_{m-1}x^m \\ &= a_1 + 2a_2x + \sum_{m=2}^{\infty}[(m+1)a_{m+1} - (m-1)a_{m-1}]x^m. \end{aligned}$$

Thus we have

$$a_1 + (2a_2 - 2a_0)x + \sum_{m=2}^{\infty}[(m+1)a_{m+1} - (m-1)a_{m-1} - 2a_{m-1}]x^m = 0,$$

that is,

$$a_1 + 2(a_2 - a_0)x + \sum_{m=2}^{\infty}(m+1)(a_{m+1} - a_{m-1})x^m = 0.$$

By the property of VAC,

$$a_1 = 0, \quad a_2 = a_0, \quad a_{m+1} = a_{m-1}, \quad m = 2, 3, \ldots.$$

This implies that

$$a_1 = a_3 = a_5 = \cdots = 0, \quad a_0 = a_2 = a_4 = \cdots .$$

Hence

$$\begin{aligned} y &= a_0 + a_0 x^2 + a_0 x^4 + \cdots \\ &= a_0 \left(1 + x^2 + x^4 + \cdots\right) \\ &= a_0 \left[\sum_{m=0}^{\infty} \left(x^2\right)^m\right] \\ &= \frac{a_0}{1 - x^2}. \end{aligned}$$

(b) (6 marks) Solve the following Legendre's equations. If solutions are series, list the first three non-zero terms in your final solution.

(i) $(1 - x^2)y'' - 2xy' + \frac{3}{4}y = 0$.

Solution: From $n(n+1) = \frac{3}{4}$ we get a positive solution $n = 1/2$. Hence the solution will be

$$y(x) = cy_1(x) + dy_2(x),$$

where

$$\begin{aligned} y_1(x) &= 1 - \frac{3}{4 \cdot 2!}x^2 - \frac{3 \cdot 3 \cdot 7}{4^2 \cdot 4!}x^4 - \cdots \\ y_2(x) &= x + \frac{5}{4 \cdot 3!}x^3 + \frac{5 \cdot 5 \cdot 9}{4^2 \cdot 5!}x^5 - \cdots \end{aligned}$$

(ii) $(1 - x^2)y'' - 2xy' + 6y = 0$.

Solution: From $n(n+1) = 6$ we get a positive solution $n = 2$. Hence the solution will be

$$y(x) = cy_1(x) + dy_2(x),$$

where

$$\begin{aligned} y_1(x) &= P_2(x) = \frac{1}{2}(3x^2 - 1), \\ y_2(x) &= x - \frac{4}{3!}x^3 - \frac{4 \cdot 6}{\cdot 5!}x^5 - \cdots \end{aligned}$$

(c) (6 marks) Let $f(x) = x^3$, $1 < x < 3$. Find the Fourier-Legendre expansion.

Solution: $P_n(x)$ are only defined on $-1 < x < 1$. So we need to make a linear transformation from (1,3) to (-1,1). Let $s = ax + b$. Then $-1 = a(1) + b, 1 = a(3) + b$. Then $a = 1, b = -2$, $s = x - 2$. Let

$$g(s) = f(x) = f(s + 2) = (s + 2)^3 = s^3 + 6s^2 + 12s + 8.$$

Note that

$$s^3 = \frac{2}{5}P_3(s) + \frac{3}{5}s; \quad s^2 = \frac{2}{3}P_2(s) + \frac{1}{3}.$$

We imply that

$$\begin{aligned} g(s) = \frac{2}{5}P_3(s) + \frac{3}{5}s + 6[\frac{2}{3}P_2(s) + \frac{1}{3}] + 12s + 8 &= \frac{2}{5}P_3(s) + 4P_2(s) + \frac{63}{5}s + 10 \\ &= \frac{2}{5}P_3(s) + 4P_2(s) + \frac{63}{5}P_1(s) + 10P_0(s) \\ &= \frac{2}{5}P_3(x-2) + 4P_2(x-2) + \frac{63}{5}P_1(x-2) + 10P_0(x-2) \end{aligned}$$