14. Equivalence Relations, Classes, & Partitions

- An **equivalence relation** on a set A is a relation that is reflexive, symmetric, and transitive.

Equality: $=$ is an equivalence relation on $\mathbb{R}$.

- for all $x \in \mathbb{R}$, $x = x$ $\therefore$ $=$ is reflexive.
- for all $x, y \in \mathbb{R}$, $(x=y) \rightarrow (y=x)$. $\therefore$ $=$ is symmetric
- for all $x, y, z \in \mathbb{R}$, $[(x=y) \wedge (y=z)] \rightarrow (x=z)$ $\therefore$ $=$ is transitive

Logical Equivalence:

Let $\mathcal{A}$ be the set of all compound propositions. Logical equivalence $\equiv$ is a relation on $\mathcal{A}$ given by the rule:

for all $P, Q \in \mathcal{A}$, $\quad P \equiv Q \quad$ if and only if $\quad P \leftrightarrow Q$ is a tautology.

Exercise 14.1. Prove that $\equiv$ is an equivalence relation on $\mathcal{A}$.

EQUIVALENCE RELATIONS AND EQUIVALENCE CLASSES

Given an equivalence relation $\mathcal{R}$ on A, for each element $a \in A$, we define **the equivalence class of a with respect to $\mathcal{R}$** as follows:

$$\boxed{[a]_{\mathcal{R}} = \{x \in A : a \mathcal{R} x\}}$$

= set of all elements of A which are related to a by $\mathcal{R}$

Ex With respect to the equivalence relation $=$ on the set $\mathbb{R}$, the equivalence class of $\sqrt{2}$ with respect to $=$ is

$$[\sqrt{2}]_{=} = \{x \in \mathbb{R} : \sqrt{2} = x\} = \{\sqrt{2}\}$$

In fact, for each real number $x \in \mathbb{R}$, the equivalence class of x with respect to $=$ is $[x]_{=} = \{x\}$

Question With respect to the equivalence relation $\equiv$ on the set $\mathcal{A}$ of all compound propositions, what is $[P \rightarrow P]_{\equiv}$?

* These notes are solely for the personal use of students registered in MAT1348.

Example 14.2. Let x and y be propositional variables, and let $A = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8\}$, where the elements $P_i \in A$ are the following compound propositions:

$P_1: \quad x \to y$ $\qquad P_3: \quad \neg(\neg x \vee y)$ $\qquad P_5: \quad \neg(x \to y)$ $\qquad P_7: \quad x \wedge \neg y$

$P_2: \quad x \vee y$ $\qquad P_4: \quad \neg x \vee y$ $\qquad P_6: \quad x \oplus y$ $\qquad P_8: \quad \neg(x \leftrightarrow y)$

Let $\mathcal{R}$ be a relation on the set A defined by $\quad (P_i, P_j) \in \mathcal{R} \iff P_i \equiv P_j$

Note: Because $\equiv$ is an equivalence relation on the set of *all* compound propositions, it follows that $\mathcal{R}$ is an equivalence relation on A.

Compute the equivalence class for each element of A.

$[P_1]_{\mathcal{R}} = \{P_1, P_4\}$ $\qquad [P_5]_{\mathcal{R}} = \{P_3, P_5, P_7\}$

$[P_2]_{\mathcal{R}} = \{P_2\}$ $\qquad [P_6]_{\mathcal{R}} = \{P_6, P_8\}$

$[P_3]_{\mathcal{R}} = \{P_3, P_5, P_7\}$ $\qquad [P_7]_{\mathcal{R}} = \{P_7, P_3, P_5\}$

$[P_4]_{\mathcal{R}} = \{P_1, P_4\}$ $\qquad [P_8]_{\mathcal{R}} = \{P_8, P_6\}$

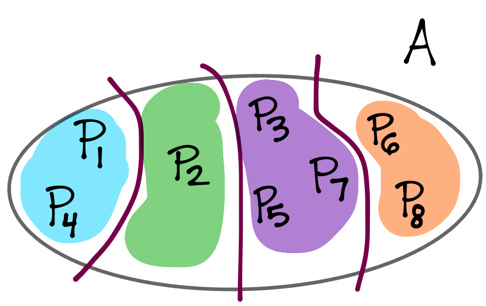
Some observations:

$[P_1]_{\mathcal{R}} = [P_4]_{\mathcal{R}}$ $\qquad [P_3]_{\mathcal{R}} = [P_5]_{\mathcal{R}} = [P_7]_{\mathcal{R}}$ $\qquad [P_6]_{\mathcal{R}} = [P_8]_{\mathcal{R}}$

$[P_1]_{\mathcal{R}} \cap [P_2]_{\mathcal{R}} = \emptyset$ $\qquad [P_2]_{\mathcal{R}} \cap [P_3]_{\mathcal{R}} = \emptyset$

$[P_1]_{\mathcal{R}} \cap [P_3]_{\mathcal{R}} = \emptyset$ $\qquad [P_2]_{\mathcal{R}} \cap [P_6]_{\mathcal{R}} = \emptyset$

$[P_1]_{\mathcal{R}} \cap [P_6]_{\mathcal{R}} = \emptyset$ $\qquad [P_3]_{\mathcal{R}} \cap [P_6]_{\mathcal{R}} = \emptyset$



$A = [P_1]_{\mathcal{R}} \cup [P_2]_{\mathcal{R}} \cup [P_3]_{\mathcal{R}} \cup [P_6]_{\mathcal{R}}$

General Observations on Equivalence Classes of an Equivalence Relation.

Let $\mathcal{R}$ be an equivalence relation on a set A. Then:

i. $a \in [a]_{\mathcal{R}}$ for all $a \in A$.

ii. $[a]_{\mathcal{R}} = [b]_{\mathcal{R}}$ if and only if $(a, b) \in \mathcal{R}$.

iii. $[a]_{\mathcal{R}} \cap [b]_{\mathcal{R}} = \emptyset$ if and only if $(a, b) \notin \mathcal{R}$.

*In fact, these properties turn out to give us what is called a **partition** of A.

CONGRUENCE MODULO n: AN EQUIVALENCE RELATION ON INTEGERS

Congruence modulo n $\equiv \pmod{n}$ — Let $n \in \mathbb{Z}^+$ be a fixed positive integer, the "modulus".

We define a relation on $\mathbb{Z}$, called **CONGRUENCE MODULO n**, denoted $\equiv \pmod{n}$, by the following rule:

$$\forall x, y \in \mathbb{Z}, \quad x \equiv y \pmod{n} \iff n \text{ divides } x-y$$

- If $x \equiv y \pmod{n}$, we say "x is congruent to y modulo n."

If the modulus n is clear from the context, then we may denote $x \equiv y \pmod{n}$ simply as $x \equiv y$.

Example 14.3. For modulus $n=2$, $x \equiv y \pmod{2} \iff 2 \mid (x-y)$
$\iff x-y$ is even

Ex $12 \equiv 4 \pmod{2}$ since 2 divides $(12-4)$.

Ex $12 \not\equiv 3 \pmod{2}$ since $2 \nmid (12-3)$

Example 14.4. $\pmod{7}$ Then $2 \equiv 16$ since $2-16=-14$ is divisible by 7.

Similarly, $0 \equiv -7 \equiv 7 \equiv 14$ $\quad 1 \equiv -6 \equiv 8 \equiv 701$ $\quad 5 \equiv 7005$

Example 14.5. EACH INTEGER m IS CONGRUENT MODULO n TO ITS REMAINDER UPON DIVISION BY n.

Let $n \in \mathbb{Z}^+$ and $m \in \mathbb{Z}$. Recall: there exist unique integers q and r such that

$$m = \underbrace{q}_{\text{quotient}}n + \underbrace{r}_{\text{remainder}} \quad \text{and} \quad 0 \le r \le n-1$$

$\therefore$ $m-r = qn$ which is divisible by n $\therefore$ $m \equiv r \pmod{n}$

Theorem 14.6. Let n be a positive integer (the modulus). Then the relation $\equiv \pmod{n}$ of congruence modulo n is an equivalence relation on $\mathbb{Z}$.

We need to prove $\equiv \pmod{n}$ is reflexive, symmetric, and transitive.

[ref.] Let $a \in \mathbb{Z}$. Then $a-a = 0 = 0 \cdot n$ $\therefore$ n divides $a-a$

$\therefore$ $a \equiv a \pmod{n}$ (by def. of $\equiv \pmod{n}$)

$\therefore$ $\equiv \pmod{n}$ is reflexive

[sym.] Let $a,b \in \mathbb{Z}$. Assume $a \equiv b \pmod{n}$. (goal: prove $b \equiv a \pmod{n}$)

Then $n \mid a-b$ $\therefore$ $\exists\, k \in \mathbb{Z}$ such that $a-b = kn$ (by def. of $\equiv \pmod{n}$)

$\Rightarrow b-a = (-k)n$ and we know $-k \in \mathbb{Z}$ since $k \in \mathbb{Z}$

$\Rightarrow n \mid b-a$ $\therefore$ $b \equiv a \pmod{n}$ $\therefore$ $\equiv \pmod{n}$ is symmetric.

[trans.] Let $a,b,c \in \mathbb{Z}$. Assume $a \equiv b$ and $b \equiv c \pmod{n}$. (goal: prove $a \equiv c \pmod{n}$)

Then $n \mid a-b$ and $n \mid b-c$ $\therefore$ $\exists\, k_1, k_2 \in \mathbb{Z}$ such that

$a-b = k_1 n$ and $b-c = k_2 n$

$$\begin{aligned} \therefore\ a-c &= a-b+b-c \\ &= k_1 n + k_2 n \\ &= (k_1+k_2)n \text{ and } k_1+k_2 \in \mathbb{Z} \text{ since } k_1, k_2 \in \mathbb{Z} \end{aligned}$$

$\therefore$ $n \mid a-c$ $\therefore$ $a \equiv c \pmod{n}$ $\therefore$ $\equiv \pmod{n}$ is transitive.

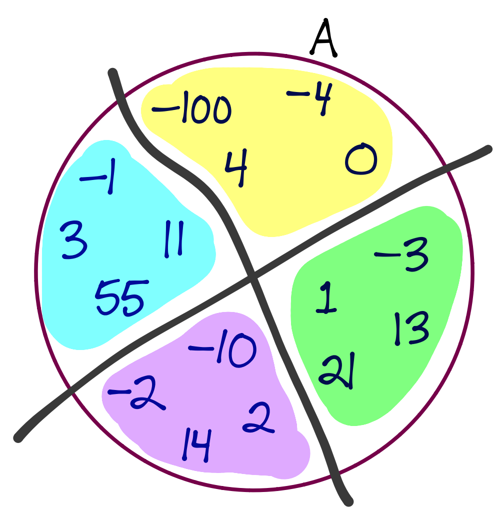
Example 14.7. Let $m = 4$ be our modulus. For each element of the following set A, compute its remainder $\pmod 4$. Determine which integers in A are congruent to each other modulo 7.

$$A = \{-100,\ -10,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 11,\ 13,\ 14,\ 21,\ 55\}$$

remainder (mod 4) 0 2 0 1 2 3 0 1 2 3 0 3 1 2 1 3

x	$=$	$km + r$
-100	$=$	$(-25)(4)+0$
-10	$=$	$(-3)(4)+2$
-4	$=$	$(-1)(4)+0$
-3	$=$	$(-1)(4)+1$
-2	$=$	$(-1)(4)+2$
-1	$=$	$(-1)(4)+3$
0	$=$	$(0)(4)+0$
1	$=$	$(0)(4)+1$

x	$=$	$km + r$
2	$=$	$(0)(4)+2$
3	$=$	$(0)(4)+3$
4	$=$	$(1)(4)+0$
11	$=$	$(2)(4)+3$
13	$=$	$(3)(4)+1$
14	$=$	$(3)(4)+2$
21	$=$	$(5)(4)+1$
55	$=$	$(13)(4)+3$



Ex The equivalence class of -100 is $[-100]_{\equiv (\text{mod} 4)} = \{-100, -4, 0, 4\}$

Ex The equivalence class of 2 is $[2]_{\equiv (\text{mod} 4)} = \{-10, -2, 2, 14\}$

PARTITIONS

A **partition** of a set A is a collection $\mathcal{P} = \{S_1, S_2, \dots\}$ of subsets $S_i \subseteq A$ such that the following three properties hold:

i. $S_i \neq \emptyset$ for all i (S_i are non-empty subsets of A)

ii. $A = S_1 \cup S_2 \cup \cdots$ (union of all S_i is all of A)

iii. $S_i \cap S_j = \emptyset$ for all $i \neq j$ (pairwise disjoint)

Example 14.8. Let $A = \{1, 2, 3, 4, 5\}$ $\mathcal{P}_1 = \{\{3,4,1\},\{2\},\{5\}\}$ is a partition of A.

$\mathcal{P}_2 = \{\{3,4\},\{2\},\{5\}\}$ is <u>not</u> a partition of A (fails property ii)

$\mathcal{P}_3 = \{\{3,4,1\},\{2\},\emptyset,\{5\}\}$ is <u>not</u> a partition of A (fails property i)

$\mathcal{P}_4 = \{\{3,4,1\},\{1,2\},\{5\}\}$ is <u>not</u> a partition of A (fails property iii)

Example 14.9. Here are two partitions of $\mathbb{Z}$:

$\mathcal{P}_1 = \{\mathbb{Z}^-, \{0\}, \mathbb{Z}^+\}$ is a partition of $\mathbb{Z}$

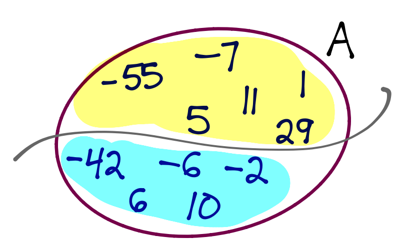
$\mathcal{P}_2 = \{\{0\},\{1,-1\},\{2,-2\},\{3,-3\},\dots\}$ is a partition of $\mathbb{Z}$

Example 14.10. Let $A = \{-55, -42, -7, -6, -2, 1, 5, 6, 10, 11, 29\}$.

i. Determine the equivalence class of -55 with respect to the *equivalence* relation $\mathcal{R}_1$ on A defined by the rule:

for all $x, y \in A$, $x \mathcal{R}_1$ if and only if $x + y$ is even. Exercise: prove $\mathcal{R}_1$ is an equivalence relation on $\mathbb{Z}$.

$[-55]_{\mathcal{R}_1} = \{-55, -7, 1, 5, 11, 29\}$



$[-42]_{\mathcal{R}_1} = \{-42, -6, -2, 6, 10\}$

Now determine the partition of A into equivalences classes with respect to $\mathcal{R}_1$.

$\mathcal{P} = \{\{-55, -7, 1, 5, 11, 29\}, \{-42, -6, -2, 6, 10\}\}$

ii. Determine the equivalence class of -55 with respect to the *equivalence* relation $\mathcal{R}_2$ on A defined by the rule:

for all $x, y \in A$, $\quad x \,\mathcal{R}_2$ if and only if $x \equiv y \pmod 9$

$[-55]_{\mathcal{R}_2} = \{-55\}$

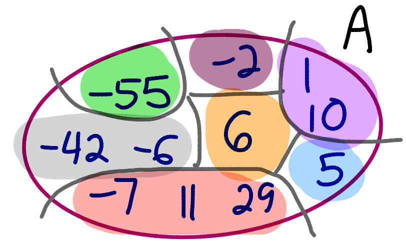
remainder(mod 9)	-55	-42	-7	-6	-2	1	5	6	10	11	29
	8	3	2	3	7	1	5	6	1	2	2

Now determine the partition of A into equivalences classes with respect to $\mathcal{R}_2$.

$[-55]_{R_2} = \{-55\}$ $\quad [-2]_{R_2} = \{-2\}$ $\quad [6]_{R_2} = \{6\}$

$[-42]_{R_2} = \{-42, -6\}$ $\quad [1]_{R_2} = \{1, 10\}$

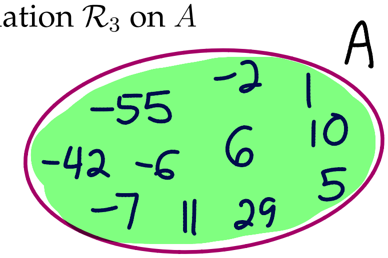
$[-7]_{R_2} = \{-7, 11, 29\}$ $\quad [5]_{R_2} = \{5\}$



iii. Determine the equivalence class of -55 with respect to the *equivalence* relation $\mathcal{R}_3$ on A defined by the rule:

for all $x, y \in A$, $\quad x \,\mathcal{R}_3$ if and only if $x \equiv y \pmod 1$

$[-55]_{\mathcal{R}_3} = \{-55, -42, -7, -6, -2, 1, 5, 6, 10, 11, 29\}$



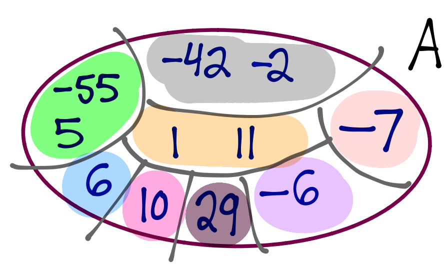
remainder(mod 1)	-55	-42	-7	-6	-2	1	5	6	10	11	29
	0	0	0	0	0	0	0	0	0	0	0

iv. Determine the equivalence class of each of the elements of A with respect to the *equivalence* relation $\mathcal{R}_4$ on A defined by the rule:

for all $x, y \in A$, $\quad x \,\mathcal{R}_4$ if and only if $x \equiv y \pmod{10}$

$[-55]_{R_4} = \{-55, 5\}$

$[6]_{R_4} = \{6\}$



remainder(mod 10)	-55	-42	-7	-6	-2	1	5	6	10	11	29
	5	8	3	4	8	1	5	6	0	1	9

Study Guide

Important terms and concepts:

equivalence relations: reflexive, symmetric, & transitive

equivalence classes: $[a]_{\mathcal{R}} = \{x \in A : x \,\mathcal{R}\, a\}$

partition $\mathcal{P} = \{S_1, S_2, \dots\}$ **of a set** A

1. $S_i \neq \emptyset$ for all i $\quad$ 2. $A = S_1 \cup S_2 \cup \cdots$ $\quad$ 3. $S_i \cap S_j = \emptyset$ for all $i \neq j$

Exercises

Sup.Ex. §7 # 1b, 2, 3, 4, 6, 8, 9, 10, 11
Rosen §9.5 # 1, 3, 7, 11, 15, 17, 25, 26, 29, 41, 47