13. Relations

Let A and B be sets. A **(binary) relation from A to B** is a subset of $A \times B$.

Let A be a set. A **(binary) relation on a set A** is a relation from A to itself.
ie a subset of $A \times A$

Example 13.1. Let $A = \{1, 2, 3\}$ and let $B = \{x, y\}$

$\mathcal{R}_1 = \{(1,x), (1,y), (3,y)\}$ $\mathcal{R}_1 \subseteq A \times B$ $\therefore \mathcal{R}_1$ is a relation from A to B

$\mathcal{R}_2 = \{(x,x), (y,x)\}$ $\mathcal{R}_2 \subseteq B \times B$ $\therefore \mathcal{R}_2$ is a relation on B

$\mathcal{R}_3 = \{(1,2), (2,1), (2,3), (3,2)\}$ $\mathcal{R}_3 \subseteq A \times A$ $\therefore \mathcal{R}_3$ is a relation on A

$\mathcal{R}_4 = \{(x,1), (y,2)\}$ $\mathcal{R}_4 \subseteq B \times A$ $\therefore \mathcal{R}_4$ is a relation from B to A

Notation for Relations

- since $(1,x) \in \mathcal{R}_1$ it means "1 is related to x by $\mathcal{R}_1$"
- since $(2,x) \notin \mathcal{R}_1$, it means "2 is not related to x by $\mathcal{R}_1$"

Special Relation Notation:

- for short, we will write $1 \mathcal{R}_1 x$ for "1 is related to x by $\mathcal{R}_1$"
  and $2 \not\mathcal{R}_1 x$ for "2 is not related to x by $\mathcal{R}_1$"

Representing Relations: a finite list of pairs of related elements VS. a rule to relate elements

For a small finite relation, we can simply write it as a list (set) of ordered related pairs of elements:
ex $\mathcal{R}_3 = \{(1,2), (2,1), (2,3), (3,2)\}$

Alternatively, some relations may be described by a rule that tells us exactly when/how two elements are related:

ex for all $u, v \in A = \{1,2,3\}$, $u \mathcal{R}_3 v$ if and only if $u+v$ is odd.

* These notes are solely for the personal use of students registered in MAT1348.

Example 13.2. Friends and Family

Ex for all people A, B, $A \mathcal{R} B$ if and only if A and B share a common ancestor.

Ex for all Facebook users X, Y, $X \mathcal{R} Y$ if and only if X and Y are "friends"

Example 13.3. Graphs

The graph of a function $f: A \to B$ is is the set of coordinate pairs $\mathcal{G} = \{(a, f(a)) : a \in A\}$

$\mathcal{G} \subseteq A \times B$ $\therefore$ $\mathcal{G}$ is a relation from A to B.

rule for $\mathcal{G}$: for all $a \in A, b \in B$, $a \mathcal{G} b$ if and only if $f(a) = b$

Ex the graph of the equation $x^2 + y^2 = 1$ is a subset of $\mathbb{R}^2$ $\therefore$ it's a relation on $\mathbb{R}$

rule: for all $x, y \in \mathbb{R}$, $x \mathcal{R} y$ if and only if $x^2 + y^2 = 1$

Example 13.4. Equality $\boxed{= \text{ is a relation on } \mathbb{R}}$

Ex. 1.8 is related to 1.8 by the relation $=$ because $1.8 = 1.8$
1.8 is not related to 1.9 by the relation $=$ because $1.8 \neq 1.9$

Properties of $=$ as a relation on $\mathbb{R}$:

① for all $x \in \mathbb{R}$, $x = x$. ② for all $x, y \in \mathbb{R}$, $(x = y) \to (y = x)$ ③ for all $x, y, z \in \mathbb{R}$, $[(x = y) \wedge (y = z)] \to (x = z)$

Example 13.5. Inequality

$\boxed{< \text{ is a relation on } \mathbb{R}}$

Ex 2 is related to 2.1 because $2 < 2.1$
2 is not related to 2 because $2 < 2$ is false.

$\boxed{\leq \text{ is a relation on } \mathbb{R}}$

Ex 2 is related to 2 because $2 \leq 2$
2 is related to 1000.1 because $2 \leq 1000.1$
2 is not related to 0 because $2 \nleq 0$

Example 13.6. Divides $\boxed{\mid \text{ ("divides") is a relation on } \mathbb{Z}}$

for all $m, n \in \mathbb{Z}$, $m \neq 0$, $m \mid n$ if and only if $n = km$ for some integer k.

Ex $5 \mid 100$ $100 \nmid 5$ $3 \mid 99$ ← these are not fractions! Never forget: "divides" is a relation on $\mathbb{Z}$, not the arithmatic operation of division!

Example 13.7. Logical Equivalence

$\equiv$ is a relation on the set of all compound proposions

rule: for all propositions P, Q, $P \equiv Q$ if and only if $P \leftrightarrow Q$ is a tautology

Properties of Relations on a Set

[Reflexive] A relation $\mathcal{R}$ on a set A is called **reflexive** if

the implication $(x \in A) \to ((x,x) \in \mathcal{R})$ is true.

Equivalently, $(x \in A) \to (x \mathcal{R} x)$

[Symmetric] A relation $\mathcal{R}$ on a set A is called **symmetric** if for all $x, y \in A$,

the implication $((x,y) \in \mathcal{R}) \to ((y,x) \in \mathcal{R})$ is true.

Equivalently, $(x \mathcal{R} y) \longrightarrow (y \mathcal{R} x)$

[Antisymmetric] A relation $\mathcal{R}$ on a set A is called **antisymmetric** if

for all $x, y \in A$, the implication $((x,y) \in \mathcal{R}$ and $(y,x) \in \mathcal{R}) \to (x = y)$ is true.

Equivalently, $(x \mathcal{R} y$ and $y \mathcal{R} x) \to (x = y)$

this should remind you of $(x \leq y$ and $y \leq x) \to (x = y)$

contrapositive form: $(x \neq y) \to ((x,y) \notin \mathcal{R}$ or $(y,x) \notin \mathcal{R})$

[Transitive] A relation $\mathcal{R}$ on a set A is called **transitive** if

for all $x, y, z \in A$,

the implication $((x,y) \in \mathcal{R}$ and $(y,z) \in \mathcal{R}) \to ((x,z) \in \mathcal{R})$ is true.

Equivalently, $(x \mathcal{R} y$ and $y \mathcal{R} z) \longrightarrow (x \mathcal{R} z)$

this should remind you of $(x \leq y$ and $y \leq z) \to (x \leq z)$

Equivalence Relations

A relation $\mathcal{R}$ on a set A is called **an equivalence relation** if

$\mathcal{R}$ is reflexive, symmetric, and transitive.

EXAMPLES OF RELATIONS ON A SET AND THEIR PROPERTIES

Example 13.8. Let $\mathcal{R}$ be a relation on $\mathbb{Z}$ defined by the rule

$$\text{for all } a, b \in \mathbb{Z}, \qquad (a,b) \in \mathcal{R} \quad \text{if and only if} \quad a + 5b \quad \text{is even.}$$

<u>Examples</u> $(0,0) \in \mathcal{R}$ because $0+5(0)=0$ is even

$(2,-3) \notin \mathcal{R}$ because $2+5(-3)=-13$ is not even

PROPERTIES

We can also write these as: $0\,\mathcal{R}\,0$ and $2 \not\mathcal{R} -3$

[reflexive] To prove $\mathcal{R}$ is reflexive, we must prove $(a \in \mathbb{Z}) \rightarrow (a\,\mathcal{R}\,a)$

Let $a \in \mathbb{Z}$. Then $a+5a = 6a = 2(3a)$ $\quad \because a \in \mathbb{Z}$ $\therefore 3a \in \mathbb{Z}$

$\therefore$ $a+5a$ is even. (def of even)

$\therefore a\,\mathcal{R}\,a$ (by the rule for $\mathcal{R}$) $\therefore \mathcal{R}$ is reflexive.

[symmetric] To prove $\mathcal{R}$ is symmetric, we must prove $a\,\mathcal{R}\,b \rightarrow b\,\mathcal{R}\,a$.

Let $a, b \in \mathbb{Z}$ be arbitrary elements of the set $\mathbb{Z}$.

Assume $a\,\mathcal{R}\,b$. (goal: prove $b\,\mathcal{R}\,a$.)

Then $a+5b$ is even (by the rule for $\mathcal{R}$) $\therefore a+5b = 2k$ for some $k \in \mathbb{Z}$

$\Rightarrow b+5a = b+5(2k-5b) = 10k-24b = 2(5k-12b)$. $\because k,b \in \mathbb{Z}$ $\therefore 5k-3b \in \mathbb{Z}$

$\therefore b+5a$ is even

$\Rightarrow b\,\mathcal{R}\,a$ (by the rule for $\mathcal{R}$) $\therefore \mathcal{R}_2$ is symmetric

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[antisymmetric] To prove $\mathcal{R}$ is antisymmetric, we must prove

$$[(a\mathcal{R}\,b) \wedge (b\mathcal{R}\,a)] \rightarrow [a=b] \text{ for all } a,b \in \mathbb{Z}$$

Wait! This is not true for all $a, b \in \mathbb{Z}$

counterexample: $3,7 \in \mathbb{Z}$ $\quad$ $3+5(7)$ is even and $7+5(3)$ is even

$\Rightarrow$ both $(3,7) \in \mathcal{R}$ and $(7,3) \in \mathcal{R}$, but $3 \neq 7$. $\therefore \mathcal{R}$ is <u>not</u> antisymmetric

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[transitive] To prove $\mathcal{R}$ is transitive, we must prove $[(a\mathcal{R}\,b) \wedge (b\mathcal{R}\,c)] \rightarrow [a\,\mathcal{R}\,c]$

Let $a,b,c \in \mathbb{Z}$ be arbitrary elements of the set $\mathbb{Z}$.

Assume $a\,\mathcal{R}\,b$ and $b\,\mathcal{R}\,c$. (goal is to prove $a\,\mathcal{R}\,c$) ← EXERCISE Prove this!

Since $\mathcal{R}$ is reflexive, symmetric and (as you will prove) transitive,

$\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.

Example 13.9. Let $\mathcal{R}$ be a relation on $\mathbb{Z}$ defined by $\mathcal{R} = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : m|n\}$

Recall: for $m,n \in \mathbb{Z}$, $m \neq 0$, "m divides n" $\Leftrightarrow$ $n = km$ for some $k \in \mathbb{Z}$

[transitive] we must prove $x\mathcal{R}y \wedge y\mathcal{R}z \to x\mathcal{R}z$

ie $[(x|y) \wedge (y|z)] \to [x|z]$

Let $x,y,z \in \mathbb{Z}$.

Assume $x|y$ and $y|z$ (goal: prove $x|z$)

Then $y = kx$ and $z = \ell y$ for some integers $k, \ell \in \mathbb{Z}$ $(x \neq 0, y \neq 0)$ (def of divides)

Consequently, $z = \ell(kx) = (\ell k)x = jx$ where $j = \ell k$

Since $\ell, k \in \mathbb{Z}$, it follows that $j \in \mathbb{Z}$.

$\therefore$ $x|z$ (def of divides)

we proved $x|y$ and $y|z \to x|z$ $\therefore$ "divides" is transitive

⋄ $\mathcal{R}$ is transitive, but not reflexive, nor symmetric, nor antisymmetric. (verify this!)

Example 13.10. Let $\mathcal{R}$ be a relation on $\mathbb{Z}^+$ defined by $\mathcal{R} = \{(m,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : m|n\}$

(same rule as in Ex. 13.9 but different set)

[antisymmetric] we must prove $(x\mathcal{R}y \wedge y\mathcal{R}x) \to (x = y)$

ie $[(x|y) \wedge (y|x)] \to [x = y]$

Let $x,y \in \mathbb{Z}^+$

Assume $x|y$ and $y|x$ (goal: prove $x = y$)

Then $y = kx$ and $x = \ell y$ for some integers $k, \ell \in \mathbb{Z}$ (def. of divides)

Note: since $x,y \in \mathbb{Z}^+$, it follows that $k, \ell \in \mathbb{Z}^+$.

Thus $y = k(\ell y) \Rightarrow y - k\ell y = 0$

$\Rightarrow y(1 - k\ell) = 0$

$\Rightarrow \cancel{y = 0}$ or $k\ell = 1$

reject because $y \in \mathbb{Z}^+$

$\therefore$ $k = \ell = 1$ because the only integer factors of 1 are ± 1 but since $k, \ell \in \mathbb{Z}^+$, the only option is $k = \ell = 1$.

$\therefore$ $y = kx \Rightarrow y = 1 \cdot x = x$ $\therefore$ $y = x$

we proved $x|y$ and $y|x \to x = y$ $\therefore$ $\mathcal{R}$ is antisymmetric as a relation on $\mathbb{Z}^+$.

⋄ As a relation on $\mathbb{Z}^+$ (instead of on $\mathbb{Z}$), $\mathcal{R}$ is reflexive, antisymmetric, and transitive, but not symmetric. (verify this!)

Example 13.11. The relation $<$ on $\mathbb{R}$:

Is $<$ reflexive? No! Counterexample: $\sqrt{2} \in \mathbb{R}$ but $\sqrt{2} < \sqrt{2}$ is false.

$\therefore$ the implication $(x \in \mathbb{R}) \to (x < x)$ is not always true. $\therefore$ $<$ is not reflexive.

Is $<$ antisymmetric? Yes! <u>proof</u> Let $x, y \in \mathbb{R}$. Then there are 3 cases for x and y:

<u>Case 1</u> Assume $x = y$. Then $x < y$ is False, thus $[(x<y) \wedge (y<x)] \to [x=y]$ is vacuously true.

<u>Case 2</u> Assume $x < y$. Then $y < x$ is False, thus $[(x<y) \wedge (y<x)] \to [x=y]$ is vacuously true.

<u>Case 1</u> Assume $x > y$. Then $x < y$ is False, thus $[(x<y) \wedge (y<x)] \to [x=y]$ is vacuously true.

$\therefore$ for all $x, y \in \mathbb{R}$, the implication $[(x<y) \wedge (y<x)] \to [x=y]$ is true. $\therefore$ $<$ is antisymmetric.

$\diamond$ $<$ is transitive and antisymmetric, but not reflexive, nor symmetric. (verify this!)

Example 13.12. Let $\mathcal{R}$ be a relation on $A = \{1, 2, 3, 4\}$ defined by

$$\mathcal{R} = \{(1,1), (2,2), (2,3), (3,2), (3,3), (4,4)\}$$

Is $\mathcal{R}$ reflexive?

Yes! <u>proof</u>: $(1,1), (2,2), (3,3), (4,4) \in \mathcal{R}$ $\therefore$ $(a \in A) \to (a \mathcal{R} a)$ is true

Is $\mathcal{R}$ antisymmetric?

<u>No</u>. Counterexample: $(2,3) \in \mathcal{R}$ and $(3,2) \in \mathcal{R}$ but $2 \neq 3$

Thus, it is not the case that $\big((a,b) \in \mathcal{R} \wedge (b,a) \in \mathcal{R}\big) \to (a = b)$ for all $a, b \in A$.

$\therefore$ $\mathcal{R}$ is <u>not</u> antisymmetric.

$\diamond$ $\mathcal{R}$ is $\underbrace{\text{reflexive, symmetric, and transitive}}$, but not antisymmetric. (verify this!)

In particular, $\mathcal{R}$ is an equivalence relation on $\{1,2,3,4\}$

Study Guide

Important terms and concepts:

☐ relation from A to B $\mathcal{R} \subseteq A \times B$	☐ relation on A $\mathcal{R} \subseteq A \times A$	x is related to y by $\mathcal{R}$ $(x, y) \in \mathcal{R}$ $\quad$ $x \, \mathcal{R} \, y$
important relations on $\mathbb{Z}$ $=$ $\leq$ $<$ $\mid$	**properties of relations:**	reflexive symmetric antisymmetric transitive

Exercises

Sup.Ex. §6 # 1, 2, 3, 5, 6, 7, 9

Rosen §9.1 # 1, 3, 5, 6, 7, 8, 9, 10, 44, 46acdf