12. Functions

Recall:

- A function $f : A \to B$ is called **injective** or "1-1" if

for all $a_1, a_2 \in A$, the implication $(f(a_1) = f(a_2)) \longrightarrow (a_1 = a_2)$ is true.

- A function $f : A \to B$ is called **surjective** or "onto" if

for all $b \in B$, there is at least one $a \in A$ such that $f(a) = b$ <u>ie</u> $f^{-1}(b) \neq \emptyset$.

The properties "injective" and "surjective" are independent properties.
Any combination of these two properties is possible.

Consider the following functions from $\mathbb{Z}$ to $\mathbb{Z}$

$id_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$
$id_{\mathbb{Z}}(k) = k$

↑ $id_{\mathbb{Z}}$ is both injective and surjective

$s : \mathbb{Z} \to \mathbb{Z}$
$s(k) = k+1$

↑ s is both injective and surjective

* <u>Note</u> the "floor function"

$\lfloor\ \rfloor : \mathbb{R} \to \mathbb{Z}$ is defined by

$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$

<u>Ex</u> $\lfloor 1.8 \rfloor = 1$ <u>Ex</u> $\lfloor -5 \rfloor = -5$

<u>Ex</u> $\lfloor -1.8 \rfloor = -2$ <u>Ex</u> $\lfloor 1.999 \rfloor = 1$

$g : \mathbb{Z} \to \mathbb{Z}$
$g(k) = \lfloor k/2 \rfloor$

↑ g is surjective but <u>not</u> injective

$h : \mathbb{Z} \to \mathbb{Z}$
$h(k) = k^3$

↑ h is injective but <u>not</u> surjective

$f : \mathbb{Z} \to \mathbb{Z}$
$f(k) = 55$

↑ f is neither injective nor surjective

Exercise give proofs or counterexamples for each of the above functions and properties

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* These notes are solely for the personal use of students registered in MAT1348.

A function $f : A \to B$ is called a **bijection** if

f is both injective and surjective.

Example 12.1. Let $g : \mathbb{R}^- \times \mathbb{R}^+ \to \mathbb{R}^- \times \mathbb{R}^-$ be the function defined as follows:

$$g(x, y) = \left(\frac{x}{y}, 3xy\right)$$

Note: since $x \in \mathbb{R}^-$ and $y \in \mathbb{R}^+$ it follows that $x/y \in \mathbb{R}^-$ and $3xy \in \mathbb{R}^-$

Prove that g is a bijection. → we must prove that g is both injective and surjective

Recall: $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$

[Surjective]. Let $(r, s) \in \mathbb{R}^- \times \mathbb{R}^-$ be an arbitrary element of g's codomain.

(goal: prove that there exists some $(x, y) \in \mathbb{R}^- \times \mathbb{R}^+$ such that $g(x, y) = (r, s)$)

We must reverse-engineer what (x, y) need to equal in terms of (r, s)

$\Rightarrow$ we need $(x, y) \in$ g's domain such that $\left(\frac{x}{y}, 3xy\right) = (r, s)$

$\Rightarrow$ we need $\frac{x}{y} = r$ and $\underbrace{3xy = s}_{①}$

$\Rightarrow \underbrace{x = ry}_{②}$ and $\underbrace{3(ry)y = s}_{\text{plug ② into ①}} \Rightarrow y^2 = \underbrace{\frac{s}{3r}}$ $\Rightarrow \underbrace{y = \sqrt{\frac{s}{3r}}}_{③}$ or $\cancel{y = -\sqrt{\frac{s}{3r}}}$ (reject because $y \in \mathbb{R}^+$)

note: $\frac{s}{3r} \in \mathbb{R}^+$ because $r, s \in \mathbb{R}^-$.

$\Rightarrow \underbrace{x = r\sqrt{\frac{s}{3r}}}_{\text{plug ③ into ②}}$

Since $r, s \in \mathbb{R}^-$, it follows that $x = r\sqrt{\frac{s}{3r}} \in \mathbb{R}^-$ and $y = \sqrt{\frac{s}{3r}} \in \mathbb{R}^+$

$\therefore (x, y) \in \mathbb{R}^- \times \mathbb{R}^+$ (g's domain).

Moreover,

$$g(x, y) = g\left(r\sqrt{\frac{s}{3r}}, \sqrt{\frac{s}{3r}}\right) = \left(\frac{r\sqrt{\frac{s}{3r}}}{\sqrt{\frac{s}{3r}}}, 3r\sqrt{\frac{s}{3r}} \cdot \sqrt{\frac{s}{3r}}\right) = (r, s).$$

$\therefore$ g is surjective

[injective]. Let $(a,b),(c,d)\in\mathbb{R}^-\times\mathbb{R}^+$ be arbitrary elements of g's domain.

Assume $g(a,b)=g(c,d)$. (goal: prove $(a,b)=(c,d)$)

Then $\left(\frac{a}{b},3ab\right)=\left(\frac{c}{d},3cd\right)$ (by def of g)

$$\Rightarrow \frac{a}{b}=\frac{c}{d} \text{ and } 3ab=3cd$$

$$\Rightarrow \underbrace{a=\frac{cb}{d}}_{①} \quad \Rightarrow 3\left(\frac{cb}{d}\right)b=3cd$$

$$\Rightarrow b^2=d^2$$

$$\Rightarrow b=d \text{ or } \cancel{b=-d}$$

← since $b,d\in\mathbb{R}^+$, they cannot have opposite signs

$$\therefore \underbrace{b=d}_{②} \quad \Rightarrow a=\frac{cb}{b} \text{ (plug ② into ①)} \quad \Rightarrow a=c$$

∴ we proved that $(g(a,b)=g(c,d))\rightarrow((a,b)=(c,d))$ ∴ g is injective.

Since g is both surjective and injective, it's a bijection. ∎

The identity function.

Let A be any set.

The **identity function on** A, denoted id_A, is the function $\mathrm{id}_A : A \to A$ defined by

$$\mathrm{id}_A(x)=x \quad \text{for all } x\in A.$$

In particular, id_A is a bijection from the set A to itself.

COMPOSITIONS OF FUNCTIONS

Let $f : A \to B$ and let $g : C \to D$ be functions.

Provided $f(A) \subseteq C$, then the **composition** g of f, denoted $g \circ f$, is the function $g \circ f : A \to D$ defined by

$$(g\circ f)(a)=g(f(a)) \quad \text{for all } a\in A.$$

Note: for all $a\in A$, we need $f(a)$ to be an element of g's domain; otherwise, $g(f(a))$ would not be defined.

Example 12.2. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 5x - 7$ and $g(x) = x^2$.

Find $g \circ f$.

f's domain

g's codomain

$$g\circ f : \mathbb{R} \to \mathbb{R}$$

$$\begin{aligned} g\circ f(x) &= g(f(x)) \\ &= g(5x-7) \\ &= (5x-7)^2 \\ &= 25x^2 - 70x + 49 \end{aligned}$$

Find $f \circ g$.

g's domain

f's codomain

$$f\circ g : \mathbb{R} \to \mathbb{R}$$

$$\begin{aligned} f\circ g(x) &= f(g(x)) \\ &= f(x^2) \\ &= 5(x^2) - 7 \\ &= 5x^2 - 7 \end{aligned}$$

Note. In general, $f \circ g \neq g \circ f$, even if both compositions are defined.

INVERSE FUNCTIONS

Let $f: A \to B$ be a function.

The <u>inverse</u> of f (if it exists) is the function $f^{-1}: B \to A$ such that

$$\boxed{f^{-1}\circ f = id_A \quad \text{and} \quad f\circ f^{-1} = id_B}$$

Equivalently,

the <u>inverse</u> of f (if it exists) is the function $f^{-1}: B \to A$ such that

for all $a \in A$, $b \in B$, $\boxed{f^{-1}(b) = a \text{ if and only if } f(a) = b.}$

Some facts about inverse functions

- Not every function has an inverse.
- If a function $f: A \to B$ has an inverse, then we call f invertible.
- If f is invertible, then its inverse is <u>unique</u> meaning there is one and <u>only one</u> function from B to A whose compositions with f give the respective identity functions.
- <u>Theorem</u> Let $f: A \to B$ be a function.
  Then f is invertible if and only if f is a bijection.

Example 12.3. For the function $g : \mathbb{R}^- \times \mathbb{R}^+ \to \mathbb{R}^- \times \mathbb{R}^-$, defined by $g(x,y) = \left(\frac{x}{y}, 3xy\right)$, verify that g's inverse $g^{-1} : \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R}^- \times \mathbb{R}^+$ is given by the rule

$$g^{-1}(r,s) = \left(r\sqrt{\frac{s}{3r}}, \sqrt{\frac{s}{3r}}\right)$$

Let $(r,s) \in \mathbb{R}^- \times \mathbb{R}^-$

$$\begin{aligned}(g \circ g^{-1})(r,s) &= g(g^{-1}(r,s)) \\ &= g\left(r\sqrt{\frac{s}{3r}}, \sqrt{\frac{s}{3r}}\right) \\ &= \left(\frac{r\sqrt{\frac{s}{3r}}}{\sqrt{\frac{s}{3r}}}, 3 \cdot r\sqrt{\frac{s}{3r}} \cdot \sqrt{\frac{s}{3r}}\right) \\ &= (r,s) \checkmark\end{aligned}$$

$\therefore g \circ g^{-1} = id_{\mathbb{R}^- \times \mathbb{R}^-}$

Let $(x,y) \in \mathbb{R}^- \times \mathbb{R}^+$

$$\begin{aligned}(g^{-1} \circ g)(x,y) &= g^{-1}(g(x,y)) \\ &= g^{-1}\left(\frac{x}{y}, 3xy\right) \\ &= \left(\frac{x}{y}\sqrt{\frac{3xy}{3\left(\frac{x}{y}\right)}}, \sqrt{\frac{3xy}{3\left(\frac{x}{y}\right)}}\right) \\ &= \left(\frac{x}{y}\sqrt{y^2}, \sqrt{y^2}\right) \\ &= (x,y) \checkmark\end{aligned}$$

* because $y \in \mathbb{R}^+$ we know $y = \sqrt{y^2}$

$\therefore g^{-1} \circ g = id_{\mathbb{R}^- \times \mathbb{R}^+}$

CARDINALITIES OF INFINITE SETS

Note. If A and B are finite sets and $f : A \to B$ is a bijection, then $|A| = |B|$.

For infinite sets, the way we compare their cardinality is through bijections. We define the notion of equality of cardinalities of infinite sets as follows:

$$|A| = |B| \quad \text{if and only if} \quad \text{there exists a bijection from } A \text{ to } B.$$

$$\boxed{\begin{array}{ccc}(\text{injective}) & \land & (\text{surjective}) \\ \downarrow & & \downarrow \\ (|A| \leq |B|) & \land & (|B| \leq |A|)\end{array}}$$

An infinite set S is called **countable** if $|S| = |\mathbb{N}|$.

Ex. $f : \mathbb{Z} \to \mathbb{N}$ defined by $f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ 2|n|-1 & \text{if } n < 0 \end{cases}$ is 1-1 and onto (verify this!)

$\therefore$ there is a bijection from $\mathbb{Z}$ to $\mathbb{N}$ $\therefore |\mathbb{Z}| = |\mathbb{N}|$

$$f\left(\begin{array}{cccccccccc} \mathbb{Z} & 0 & -1 & 1 & -2 & 2 & -3 & 3 & -4 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\ \mathbb{N} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \end{array}\right.$$

Fact. There is no bijection from $\mathbb{R}$ to $\mathbb{N}$. Therefore, the set of real numbers is called **uncountable.**

STUDY GUIDE

Important terms and concepts:

☐ bijection
injective & surjective

☐ identity function
for all $x \in A$, $\mathrm{id}_A(x) = x$

☐ composition
$(f \circ g)(x) = f(g(x))$

☐ inverse of $g : A \to B$
$g^{-1} \circ g = \mathrm{id}_A$
$g \circ g^{-1} = \mathrm{id}_B$

Exercises	Sup.Ex. §5 # 1, 2, 3, 4, 5, 8, 10, 11 Rosen §2.3 # 1, 9, 10, 11, 12, 13, 14, 15, 33, 34, 35, 36, 37, 38, 71