10. Set Operations & Set Identities

Basics of Set Theory:

- ☐ set element when two sets are equal
- ☐ describing a set:
  - set-builder notation
  - list notation (order / multiplicity do not affect an element's membership in a set)
- ☐ when two sets are equal subset proper subset
- ☐ empty set $\emptyset$ universal set $\mathcal{U}$
- ☐ cardinality of a finite set S: $|S|$ power set of a set S: $\mathcal{P}(S)$

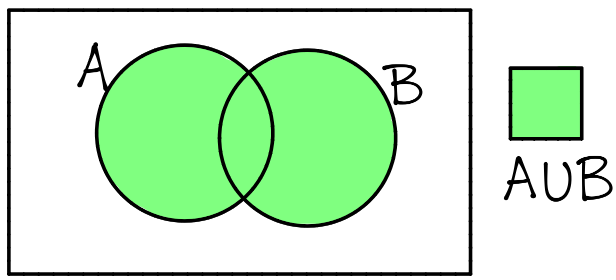
SETS OPERATIONS

Venn diagrams are a graphical method for depicting sets and set operations.

Union.

The union of sets A and B, denoted $A \cup B$, is the set

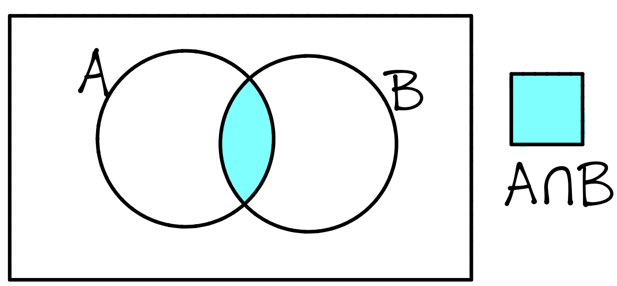
$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$



Intersection.

The intersection of sets A and B, denoted $A \cap B$, is the set

$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$$

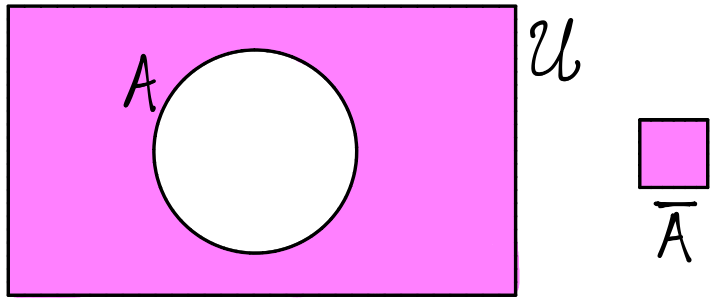


* These notes are solely for the personal use of students registered in MAT1348.

Complement.

The complement of a set A, denoted $\overline{A}$, is the set

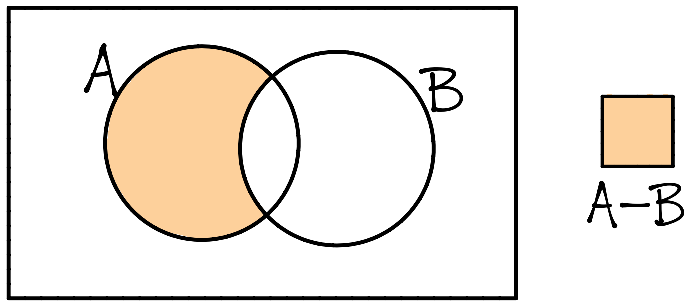
$$\overline{A} = \{x : (x \in \mathcal{U}) \wedge (x \notin A)\}$$



Difference.

The difference "A minus B," denoted $A-B$, is the set

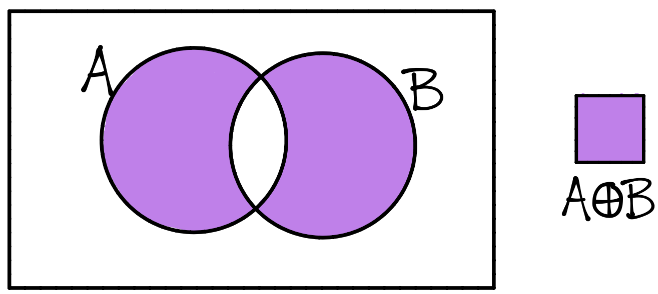
$$A-B = \{x : (x \in A) \wedge (x \notin B)\}$$



Symmetric Difference.

The symmetric difference of sets A and B, denoted $A \oplus B$, is the set

$$A \oplus B = \{x : (x \in A) \oplus (x \in B)\}$$



Example 10.1. Let $A = \{1, 2, 3\}$ $\quad B = \{1, 3, 5, 7\}$ $\quad C = \{5, 7\}$ be subsets of a universal set $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A \cup B = \{1,2,3,5,7\}$

$A \cap B = \{1,3\}$

$A \cap C = \{\} = \emptyset$

$B \cap C = \{5,7\} = C$

$B \cup C = \{1,3,5,7\} = B$

$\overline{B} = \{2,4,6,8\}$

$\overline{A} = \{4,5,6,7,8\}$

$A - B = \{2\}$

$B - C = \{1,3\}$

$A \oplus B = \{2,5,7\}$

$B \oplus C = \{1,3\}$

Disjoint Sets.

Sets A and B are called **disjoint** if $A \cap B = \emptyset$.

Ex. A and C are disjoint.
(from above example)

Set Identities

A **set identity** is an equation involving sets and set operations that is true *no matter what* particular sets we consider.

Example 10.2. For all sets A, B, C, the following equation is true:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

look familiar?
▶ distributive law!

Membership Tables

We can verify a set identity using a **membership table**.

- Membership tables are similar to truth tables, but more like attendance sheets.
- If there are n sets involved in an identity, then the table will have 2^n rows.
- Each row corresponds to one possible "location" of an element $x \in \mathcal{U}$, *relative to* the sets in the identity.

Example 10.3. Using a membership table, prove $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (De Morgan)

A	B	$A \cup B$	$\overline{A \cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

Since membership is the same for $\overline{A \cup B}$ and $\overline{A} \cap \overline{B}$ it follows that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

↑
for 2 sets A and B, these 4 rows cover all possible cases for the membership of an arbitrary element x of the universal set relative to A and B.

Using the Table of Set Identities

- As with logical equivalences, we can verify a set identity using established laws from the **Table of Important Set Identities**

Example 10.4. Let A, B, and C be subsets of a universal set $\mathcal{U}$.

Using set identities, prove that $\overline{(B \cup C) - A} = (\overline{C} \cap \overline{B}) \cup A$

$\overline{(B \cup C) - A} = \overline{(B \cup C) \cap \overline{A}}$ (Difference)

$= \overline{(B \cup C)} \cup \overline{\overline{A}}$ (DeMorgan's)

$= \overline{(B \cup C)} \cup A$ (Double Complementation)

$= (\overline{B} \cap \overline{C}) \cup A$ (DeMorgan's)

$= (\overline{C} \cap \overline{B}) \cup A$ (Commutative)

$\therefore \overline{(B \cup C) - A} = (\overline{C} \cap \overline{B}) \cup A$

Rigorous Proofs Involving Sets

Let A and B be subsets of a universal set $\mathcal{U}$.
To give a **rigourous proof that** $A \subseteq B$ we must prove: *for all* $x \in \mathcal{U}$*, the implication* $(x \in A) \to (x \in B)$ *is true.* We can prove this **directly** as follows:

- Let $x \in \mathcal{U}$ be an arbitrary element of the universal set. Assume $x \in A$.
- Then, step-by-step, prove that $x \in B$ must be true.

We can also prove the implication $(x \in A) \to (x \in B)$ using an **indirect proof**, a **proof by contradiction**, a **proof by cases**, or some suitable combination of proof strategies.

Example 10.5. Prove the following theorem:

Theorem 10.5. Let A and B be subsets of the universal set.

$$\text{Then } \underbrace{A - B = \emptyset}_{P} \text{ if and only if } \underbrace{A \subseteq B}_{Q}.$$

$$P \longleftrightarrow Q$$

We must prove $P \to Q$ and $Q \to P$ (i.e. give a proof of equivalence)

<u>proof</u>. Let A and B be sets.

($\Longrightarrow$) We will give an indirect proof of $P \to Q$.

Assume $A \nsubseteq B$ (ie assume $\neg Q$ is true) (goal: prove $A - B \neq \emptyset$ ie prove $\neg P$)

Then it is not the case that (for all $x \in \mathcal{U}$, $(x \in A) \to (x \in B)$) (by def of $A \nsubseteq B$)

$\therefore$ there exists at least one element $x \in \mathcal{U}$ such that $x \in A$ but $x \notin B$.

$\therefore$ $x \in A - B$ (by def. of $A - B$)

$\therefore$ $A - B \neq \emptyset$ because $A - B$ contains at least one element, namely x.

We proved $\neg Q \to \neg P$ $\therefore$ $P \to Q$ is true.

($\Longleftarrow$) We will give an indirect proof of $Q \to P$

Assume $A - B \neq \emptyset$ (i.e. assume $\neg P$) (goal: prove $A \nsubseteq B$ ie prove $\neg Q$)

Then there must exist at least one element $x \in A - B$ (since $A - B \neq \emptyset$)

Then $x \in A$ and $x \notin B$ (by definition of set difference $A - B$)

$\therefore$ $A \nsubseteq B$ (goal!) We proved $\neg P \to \neg Q$ $\therefore$ $Q \to P$ is true.

Since $P \to Q$ is true and $Q \to P$ is true, it follows that $P \leftrightarrow Q$ is true. $\blacksquare$

Exercise 10.6. Prove or disprove each of the following claims. If you prove it is true, give a rigorous proof; otherwise, give a *counterexample*, consisting of concrete sets and a brief explanation why they demonstrate that the claim can be false.

Claim 1. Let A, B, and C be sets. If $A - B = A - C$, then $B = C$.

this can be false. Counterexample: Let $A=\{1\}$, $B=\{2\}$, $C=\{3\}$.

Then $A-B=\{1\}$ $\quad A-C=\{1\}$ $\therefore$ $A-B=A-C$.

But $B \neq C$ since these two sets do not contain the same elements.

Claim 2. Let A, B, and C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. True!

<u>proof</u>. Let $A, B,$ and C be sets. Assume $A \subseteq B$ and $B \subseteq C$. (goal: prove $A \subseteq C$)

Let $x \in \mathcal{U}$. Assume $x \in A$ (goal: prove $x \in C$),

this sets up a rigorous proof that $A \subseteq C$ to achieve this goal)

Then $x \in B$ since $A \subseteq B$ means $(x \in A) \to (x \in B)$ is true for all $x \in \mathcal{U}$.
Consequently, $x \in C$ since $B \subseteq C$ means $(x \in B) \to (x \in C)$ for all $x \in \mathcal{U}$. $\therefore$ $(x \in A) \to (x \in C)$

We proved $(A \subseteq B \ \wedge\ B \subseteq C) \to (A \subseteq C)$ $\blacksquare$

STUDY GUIDE

Basic terms and concepts of Set Theory:

☐ set S ☐ element $x \in S$ ☐ subset $T \subseteq S$ ☐ proper subset $T \subset S$ ☐ equality $S = T$ ☐ cardinality $|S|$

Some important sets:

☐ empty set $\emptyset$ ☐ universal set $\mathcal{U}$ ☐ naturals $\mathbb{N}$ ☐ integers $\mathbb{Z}$ $\mathbb{Z}^-$ $\mathbb{Z}^+$ ☐ rationals $\mathbb{Q}$ $\mathbb{Q}^-$ $\mathbb{Q}^+$ ☐ reals $\mathbb{R}$ $\mathbb{R}^-$ $\mathbb{R}^+$

Building new sets from old:

☐ power set of S $\mathcal{P}(S)$ ☐ Cartesian product of two (or more) sets $S \times T$ $\quad S_1 \times S_2 \times \cdots \times S_t$

Set Operations:

☐ union $S \cup T$ ☐ intersection $S \cap T$ ☐ complement $\overline{S}$ ☐ difference $S - T$ ☐ symmetric difference $S \oplus T$

Set identities:

☐ verify using membership tables ☐ verify using a rigorous proof

☐ prove other identities using the laws from the Table of Important Set Identities

---

Exercises Sup.Ex.	§4 # 1, 2, 3, 4, 5, 6(use a rigorous proof), 9, 11
Rosen (8th ed.)	§2.2 # 1, 3, 4, 5–13(using membership tables or rigorous proofs) 14, 15, 17, 19, 21, 23, 31, 41

Table of Important Set Identities

1. 2.	$A \cup \emptyset = A$ $A \cap \mathcal{U} = A$	Identity Laws
3. 4.	$A \cup \mathcal{U} = \mathcal{U}$ $A \cap \emptyset = \emptyset$	Domination Laws
5. 6.	$A \cup A = A$ $A \cap A = A$	Idempotent Laws
7.	$\overline{(\overline{A})} = A$	(Double) Complementation Law
8. 9.	$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
10. 11.	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
12. 13.	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Laws
14. 15.	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's Laws
16. 17.	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Laws
18. 19.	$A \cup \overline{A} = \mathcal{U}$ $A \cap \overline{A} = \emptyset$	Complement Laws
20.	$A - B = A \cap \overline{B}$	Difference Law
21. 22.	$A \oplus B = (A - B) \cup (B - A)$ $A \oplus B = (A \cup B) - (A \cap B)$	Symmetric Difference Laws