11. Functions

Building new sets from old:

☐ power set of S $\mathcal{P}(S)$

☐ Cartesian product of two (or more) sets $S \times T$ $S_1 \times S_2 \times \cdots \times S_t$

Set Operations:

☐ union $S \cup T$ ☐ intersection $S \cap T$ ☐ complement $\overline{S}$ ☐ difference $S - T$ ☐ symmetric difference $S \oplus T$

Set identities:

☐ verify using membership tables ☐ rigorous proof involving sets

☐ prove other identities using the laws from the Table of Important Set Identities

FUNCTIONS

A **function** consists of:

- a set A called the **domain**,
- a set B called the **codomain**, and
- a "rule" f that assigns to each element $a \in A$ **exactly one** element $b \in B$.

name of function $f : A \to B$ codomain of f

domain of f

We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element $a \in A$.

- for each element $a \in A$, $f(a)$ is called the image of a (in particular, $f(a) \in B$)
- for any subset $S \subseteq A$, we may consider the set $f(S) := \{f(a) : a \in S\}$

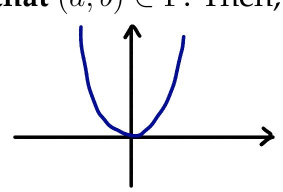
  The set $f(S)$ is called the image of S.
- In particular, the set $f(A)$ is the image of the domain, also called the range of f.
- for each $b \in B$, define the set $f^{-1}(b) := \{a \in A : f(a) = b\}$.

  the set $f^{-1}(b)$ is called the preimage of b

Alternatively, we can define functions in terms of their graphs as follows:

A **function** $f : A \to B$ can be viewed as a subset $\Gamma \subseteq A \times B$ such that **for each** $a \in A$ **there is exactly one** $b \in B$ **such that** $(a, b) \in \Gamma$. Then, for all $(a, b) \in \Gamma$, we write $b = f(a)$.

Ex. $f : \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$



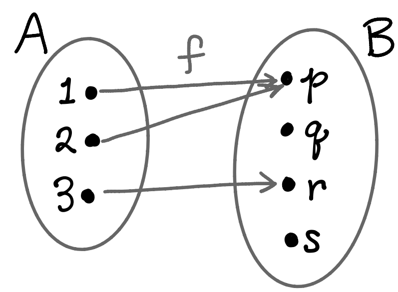
the graph of f consists of all ordered pairs $(x,y) \in \mathbb{R} \times \mathbb{R}$, such that $y = f(x)$

$\Gamma = \{(x, x^2) : x \in \mathbb{R}\}$

* These notes are solely for the personal use of students registered in MAT1348.

Example 11.1. Let $A = \{1, 2, 3\}$ and $B = \{p, q, r, s\}$

Define $f : A \to B$ as follows: $f(1) = p$
$f(2) = p$
$f(3) = r$



$f(A) = f(\{1,2,3\}) = \{p, r\}$ is the range of f

$f(\{1,2\}) = \{p\}$ is the image of the set $\{1,2\}$

f is a function from A to B.

The graph of f is the set

$\Gamma = \{(1,p), (2,p), (3,r)\}$

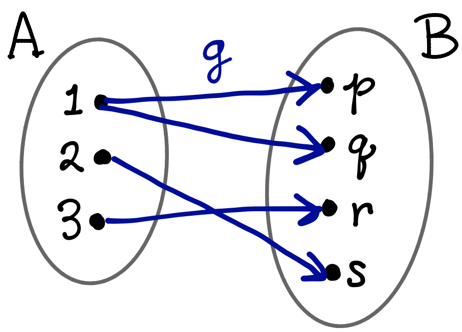
$f^{-1}(p) = \{1, 2\}$ is the preimage of p.

$f^{-1}(q) = \emptyset$ is the preimage of q.

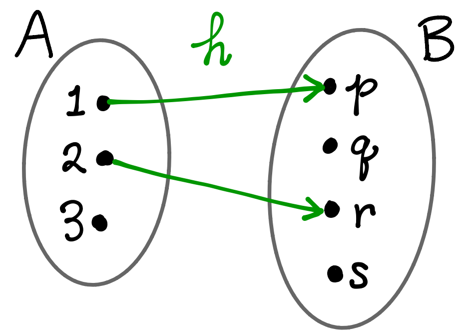
$f^{-1}(r) = \{3\}$ is the preimage of r.

$f^{-1}(s) = \emptyset$ is the preimage of s

So $\Gamma \subseteq A \times B$ and Γ contains exactly one pair for each element of A



g is not a function from A to B because $1 \in A$ is assigned to more than one element of B



h is not a function from A to B because $3 \in A$ is not assigned to any element of B.

Examples of functions given by rules (instead of arrow diagrams).

Ex. $f : \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$

Ex. $f : \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Q}$
$f(m, n) = \frac{m}{n}$

Ex. $f : \mathbb{R} \to \mathbb{Z} \times \mathbb{Q}$
$f(x) = (8, 1.2)$
(a constant function)

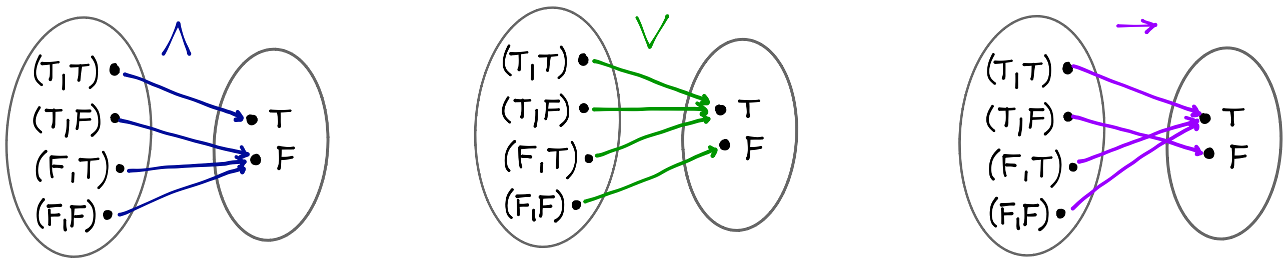
Ex. A binary operation on a set A is a function from $A \times A$ to A

Ex. Addition and multiplication of integers are binary operations on $\mathbb{Z}$:

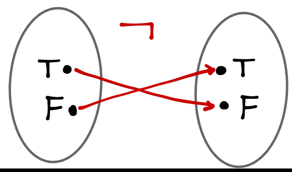
plus: $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$
plus$(a, b) = a + b$

times: $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$
times$(n, m) = nm$

More examples: domain $\{T,F\}\times\{T,F\}$ codomain $\{T,F\}$



Ex. domain $\{T,F\}$
codomain $\{T,F\}$



INJECTIVE (ONE-TO-ONE) FUNCTIONS

A function $f : A \to B$ is called **injective** or **one-to-one**, if for all $x, y \in A$, the implication

$$\Big(f(x) = f(y)\Big) \to \Big(x = y\Big) \qquad \text{is True.}$$

Equivalently (contrapositive form) for all $x,y\in A$, $\big(x\neq y\big)\to\big(f(x)\neq f(y)\big)$ is True.

Thus, each distinct element of A is assigned its own distinct unique element of B.

To rigorously prove that a function $f : A \to B$ is **injective** we must prove: *for all $a_1, a_2 \in A$, the implication $(f(a_1) = f(a_2)) \to (a_1 = a_2)$ is true.* We can prove this **directly** as follows:

- Let $a_1, a_2 \in A$. Assume $f(a_1) = f(a_2)$.
- Then, step-by-step, prove that $a_1 = a_2$ must be true.

Example 11.2. Prove that $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (2x + 1, 8x - 1)$ is injective.

proof. Let $x_1, x_2 \in \mathbb{R}$ be arbitrary elements of the domain of f.

Assume $f(x_1) = f(x_2)$ (goal: prove $x_1 = x_2$)

Then $(2x_1+1,\ 8x_1-1) = (2x_2+1,\ 8x_2-1)$ by f's rule

$\implies 2x_1+1 = 2x_2+1$ and $8x_1-1 = 8x_2-1$ since both coordinates must match to be equal

$\implies 2x_1 = 2x_2$ and $8x_1 = 8x_2$

$\implies x_1 = x_2$ and $x_1 = x_2$ (goal!)

We proved $(f(x_1)=f(x_2)) \longrightarrow (x_1 = x_2)$ $\therefore$ f is injective $\square$

Example 11.3. Is $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = x^2$ injective?

Let $a,b \in \mathbb{R}^+$ (the domain of f)

Assume $f(a)=f(b)$. Then $a^2=b^2$

$$\Rightarrow a^2-b^2=0$$

$$\Rightarrow (a-b)(a+b)=0$$

$a=b$ or $\cancel{a=-b}$

Since, $a,b \in \mathbb{R}^+$, the only possible solution is $a=b$

Note. Since, for this function, the domain is $\mathbb{R}^+$, we know that neither a nor b are negative... so $a=-b$ is not actually possible.

Thus, we proved $(f(a)=f(b)) \to (a=b)$ is True for all $a,b \in \mathbb{R}^+$.

$\therefore f$ is injective (1-1). $\square$

Example 11.4. Is $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ injective?

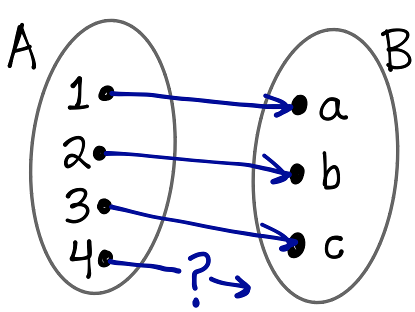
No! Counterexample: 1 and -1 are two distinct elements of the domain $\mathbb{R}$, yet $g(1)=g(-1)=1$.

$\therefore g$ is not injective (1-1).

Example 11.5. Let A and B be sets such that $|A| = 4$ and $|B| = 3$.

Does there exist an injective function $f : A \to B$?

No!



Informal explanation:

If $|A|>|B|$, then, at some point, B will "run out" of "new" distinct images for the elements of A.

Theorem 11.6. Let A and B be finite sets.
If there exists an injective function $f : A \to B$, then $|A| \le |B|$.

$\underbrace{\text{If there exists an injective function } f : A \to B}_{P} \longrightarrow \underbrace{|A| \le |B|}_{Q}$

Theorem 11.6 in contrapositive form: Let A and B be sets.

If $\underbrace{|A| > |B|}_{\neg Q}$, then $\longrightarrow$ $\underbrace{\text{there does not exist an injective function from A to B}}_{\neg P}$.

SURJECTIVE (ONTO) FUNCTIONS

A function $f : A \to B$ is called **surjective** or **onto** if, for every element $b \in B$, there exists at least one element $a \in A$ such that $f(a) = b$.

Equivalently, for all $b \in B$, $f^{-1}(b) \neq \emptyset$.

Thus, each element of the codomain B is the image of at least one element of A.

To rigorously prove that a function $f : A \to B$ is **surjective** we must prove: *for all $b \in B$, there exists at least one element $a \in A$ such that $f(a) = b$.* We can prove this **"constructively"** as follows:

- Let $b \in B$ be an arbitrary element of the codomain of f.
- Construct (figure out / reverse-engineer) an element $a \in A$, in terms of b, such that $f(a) = b$. How to do this depends on the function f, but, usually, you set $f(a) = b$, write the expression for $f(a)$ and then isolate (if possible) for a in terms of b. It can be trickier if f is not an injective function. You should also double-check that the element a you obtain is actually an element of the domain of f.

Example 11.7. Prove that $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (2x, y)$ is surjective.

Let $(a,b) \in \mathbb{R}^2$ (the codomain).

In order for $f(x,y) = (a,b)$ we need $(2x, y) = (a,b)$

$\Rightarrow 2x = a$ and $y = b$

$\Rightarrow x = \frac{a}{2}$ $(\in \mathbb{R})$ and $y = b$ $(\in \mathbb{R})$

$\Rightarrow (\frac{a}{2}, b) \in \mathbb{R} \times \mathbb{R}$ (the domain)

So for any $(a,b) \in \mathbb{R} \times \mathbb{R}$ (the codomain), we can find at least one element of the domain, namely $(\frac{a}{2}, b) \in \mathbb{R}^2$, such that $f(\frac{a}{2}, b) = (a,b)$

this is a "constructive" proof

$\therefore$ f is surjective (onto).

Example 11.8. Is $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by $f(m,n) = (2m, n)$ surjective?

No! f is not surjective (onto).

Counterexample: $(1,1)$ is an element of the codomain of f but there is no element of the domain with $(1,1)$ as its image because every element of the domain is assigned to an element of the codomain whose 1st coordinate is even.

Example 11.9. Is $q : \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Q}$ defined by $q(j,k) = \frac{j}{k}$ surjective? Is it injective?

q is surjective.

proof Let $y \in \mathbb{Q}$ be an arbitrary element of the codomain $\mathbb{Q}$.

Then, since y is a rational number, we can find integers $n, d \in \mathbb{Z}$ such that $d \neq 0$ and $y = \frac{n}{d}$. We may further assume that $d > 0$ [If $d<0$, then we can rewrite $\frac{n}{d}$ as $\frac{-n}{-d}$ and $-d>0$]

Thus, $n \in \mathbb{Z}$, $d \in \mathbb{Z}^+$ and $q(n,d) = \frac{n}{d} = y$. $\therefore$ q is surjective. $\square$

q is not injective

Counterexample: $(2,3)$ and $(10,15) \in \mathbb{Z} \times \mathbb{Z}^+$

$$f(2,3) = \frac{2}{3} = 0.\overline{6} \quad \text{and} \quad f(10,15) = \frac{10}{15} = 0.\overline{6}$$

So $f(2,3) = f(10,15)$ but $(2,3) \neq (10,15)$

$\therefore$ q is not injective.

Similar to Theorem 11.6, we have the following theorem that relates cardinality and surjective functions:

Theorem 11.10. Let A and B be finite sets.
If there exists a surjective function $f : A \to B$, then $|A| \geq |B|$.

Equivalently, (in contrapositive form):

If $|B| > |A|$, then there does not exist a surjective function from A to B.

Study Guide

Important terms and concepts:

☐ function ☐ domain ☐ codomain ☐ image ☐ preimage
☐ injective (one-to-one) function ☐ surjective (onto) function

Exercises

Sup.Ex. §5 # 1abd, 3, 4, 5, 7, 9, 10
Rosen §2.3 # 1, 4ac, 7a, 8, 9, 10, 11, 12, 13, 15