CARLETON UNIVERSITY

FINAL/DEFERRED
EXAMINATION
DECEMBER 2016

DURATION: 3 HOURS

Department and Course Number: Mathematics and Statistics, MATH 1104ABCDE
Course Instructors: Ş. Alaca (c), J. Nilsson, R. Mallick, M. Blenkinshop, M. Sadeghi

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Instructions:

1. Please circle your section below.

- Section A (Ş. Alaca)
- Section B (J. Nilsson)
- Section C (R. Mallick)
- Section D (M. Blenkinsop)
- Section E (M. Sadeghi)
- Section F (M. Sadeghi)

2. Please provide your name and student number below.

Last Name ______________________ Given Names __________________

Student Number ___________________

3. This examination contains 13 pages. Please report any missing pages to the proctor.

ANSWER ALL QUESTIONS IN PART I and PART II (pp.3-12)

Question	Maximum Mark	Mark Obtained
Part I: multiple-choice questions	36	
Part II: 1	12	
2	8	
3	8	
4	14	
5	10	
6	12	
Total	100	

Multiple-Choice Answer Sheet

1. (a) (b) (c) (d) (e)

2. (a) (b) (c) (d) (e)

3. (a) (b) (c) (d) (e)

4. (a) (b) (c) (d) (e)

5. (a) (b) (c) (d) (e)

6. (a) (b) (c) (d) (e)

7. (a) (b) (c) (d) (e)

8. (a) (b) (c) (d) (e)

9. (a) (b) (c) (d) (e)

10. (a) (b) (c) (d) (e)

11. (a) (b) (c) (d) (e)

12. (a) (b) (c) (d) (e)

PART I: Multiple Choice Questions. Three marks each. No partial marks. Circle the correct answer on the Multiple-Choice Answer Sheet on page 2. There is only one correct answer for each question.

1. Consider the following augmented matrix of a system of linear equations:

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 & 7 \\ -1 & -1 & 0 & 1 & 1 \end{array}\right].$$ The system has

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a) a unique solution

(b) infinitely many solutions with one free variable

(c) infinitely many solutions with two free variables

(d) infinitely many solutions with three free variables

(e) no solutions

2. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation such that $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. What is $T\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right)$?

(a) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 9 \\ -8 \end{bmatrix}$ **(c)** $\begin{bmatrix} 9 \\ 8 \end{bmatrix}$ **(d)** $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ **(e)** $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$$T\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) = T\left(3\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 3T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) - 2T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$
$$= 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-6 \\ 6-2 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

3. Let $A^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$. If $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the solution of the matrix equation $Ax = b$, what is x_1?

(a) -2 **(b)** 2 **(c)** 1 **(d)** -1 **(e)** $\frac{23}{2}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}b = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 5-3 \\ 10-4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

4. Let $u = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix}$ and $w = \begin{bmatrix} 3 \\ 1 \\ 5 \\ t \end{bmatrix}$.

For what value of t is the set $\{u, v, w\}$ linearly **dependent**?

(a) -3 **(b)** -1 **(c)** 3 **(d)** 2 **(e)** 0

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 0 & 1 & 5 \\ -1 & -1 & t \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 1 & 5 \\ 0 & 1 & 3+t \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & -2+t \end{bmatrix}$$

$$-2+t=0 \Rightarrow t = 2$$

5. Let A, B and C be 3×3 matrices. If $\det A = 2$, $\det B = 4$, and $\det C = 8$, what is $\det(2AB^{-1}C^T)$?

(a) 2^8 **(b)** 2^6 **(c)** 2^5 **(d)** 2^4 **(e)** 2^3

$$\det(2AB^{-1}C^T) = 2^3 \cdot \det A \cdot \det B^{-1} \cdot \det C^T$$

$$= 2^3 \cdot 2 \cdot \frac{1}{2^2} \cdot 2^3 = 2^5$$

6. Let A be a 5×8 matrix such that row echelon form has 5 pivot positions (leading entries). Which of the following statements is **FALSE**?

(a) $\dim \mathrm{Nul} A = 3$.

(b) $\mathrm{Nul} A = \mathbb{R}^3$.

(c) $\mathrm{Rank} A = 5$.

(d) $\dim \mathrm{Col} A = 5$.

(e) $\mathrm{Col} A = \mathbb{R}^5$.

7. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 & 4 \end{bmatrix}$. What is the dimension of Nul$A$?

(a) 4 **(b)** 1 **(c)** 0 **(d)** 5 **(e)** 3

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

8. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$. Find the matrix X such that $2X - B = AX + I$.

(a) $\frac{1}{2}\begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}$ **(b)** $\frac{1}{3}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ **(c)** $\frac{1}{3}\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ **(d)** $\frac{1}{2}\begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix}$ **(e)** $\frac{1}{3}\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$

$$2X - AX = I + B$$

$$\Rightarrow (2I - A)X = I + B \Rightarrow \left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\right)X = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} X = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2+2 & -2+4 \\ 0+1 & 0+2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix}$$

9. Let $A = \begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$ and $x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

You are given that x is an eigenvector of A. What is the corresponding eigenvalue?

(a) 1 **(b)** -1 **(c)** -3 **(d)** 2 **(e)** 3

$$\begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -7+4 \\ -2+5 \\ 2-2 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix} = 3\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

10. If the orthogonal projection of the vector $x = \begin{bmatrix} 6 \\ 0 \\ 9 \end{bmatrix}$ onto the vector $u = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, what is the value of b?

(a) -4 **(b)** -1 **(c)** 1 **(d)** $+4$ **(e)** 0

$$\hat{x} = \frac{x \cdot u}{u \cdot u} u = \frac{6+0+18}{1+1+4} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

11. What is the standard form $a + bi$ of the complex number $\dfrac{5+12i}{2-3i}$?

(a) $-2-3i$ **(b)** $-2+3i$ **(c)** $2+3i$ **(d)** $3-2i$ **(e)** $-3+2i$

$$\frac{5+12i}{2-3i} = \frac{(5+12i)(2+3i)}{(2-3i)(2+3i)} = \frac{10-36+(15+24)i}{4+9}$$

$$= \frac{-26+39i}{13} = -2+3i$$

12. Let $A = \begin{bmatrix} 1 & -9 \\ 4 & 1 \end{bmatrix}$. What are the eigenvalues of A?

(a) $1, 6$ **(b)** $2 \pm 4i$ **(c)** $4 \pm 2i$ **(d)** $6 \pm i$ **(e)** $1 \pm 6i$

$$\begin{vmatrix} 1-\lambda & -9 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 36 = 0$$

$$\Rightarrow (1-\lambda)^2 = -36 \Rightarrow 1-\lambda = \pm 6i$$

$$\Rightarrow \lambda = 1 \pm 6i$$

PART II: Long answer questions. Show all your work.

[12] **1.** Find the general solution of the following system of linear equations. Write the solution in vector parametric form.

$$-x_1 + 3x_2 - 2x_3 + 4x_4 = 0$$
$$2x_1 - 6x_2 + x_3 - 2x_4 = -3$$
$$x_1 - 3x_2 + 4x_3 - 8x_4 = 2$$

$$\left[\begin{array}{cccc|c} -1 & 3 & -2 & 4 & 0 \\ 2 & -6 & 1 & -2 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -3 & 4 & -8 & 2 \\ 2 & -6 & 1 & -2 & -3 \\ -1 & 3 & -2 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -3 & 4 & -8 & 2 \\ 0 & 0 & -7 & 14 & -7 \\ 0 & 0 & 2 & -4 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -3 & 4 & -8 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & -2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_2 and x_4 are free variables

$x_3 = 1 + 2x_4$

$x_1 = -2 + 3x_2$

So,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2+3x_2 \\ x_2 \\ 1+2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

[8] **2.** Let $A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 4 \\ 1 & 1 & 3 \end{bmatrix}$. Find the inverse of the matrix A.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & -2 & 3 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & -2 & 3 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -5 & 4 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

So, $A^{-1} = \begin{bmatrix} 2 & -5 & 4 \\ 1 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}$

[8] **3.** Let $A = \begin{bmatrix} 7 & 0 & 3 & 1 \\ 3 & 6 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 5 & 0 & 1 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 7 \\ 4 \\ 9 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. You are given that $\det A = 96$.

Use Cramer's Rule to find x_1 (without solving for x_2, x_3 and x_4) in the matrix equation $Ax = b$.

$$A_1(b) = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 7 & 6 & 0 & 3 \\ 4 & 1 & 0 & 2 \\ 9 & 0 & 1 & 3 \end{bmatrix}$$

cofactor expansion along the third column:

$$|A_1(b)| = 3\begin{vmatrix} 7 & 6 & 3 \\ 4 & 1 & 2 \\ 9 & 0 & 3 \end{vmatrix} - \begin{vmatrix} 3 & 0 & 1 \\ 7 & 6 & 3 \\ 4 & 1 & 2 \end{vmatrix}$$

$$= 3\left[21 + 108 + 0 - (27 + 72 + 0)\right]$$

$$- \left[36 + 0 + 7 - (24 + 0 + 9)\right]$$

$$= 3\left[129 - 99\right] - (43 - 33)$$

$$= 3(30) - 10 = 80$$

$$x_1 = \frac{|A_1(b)|}{|A|} = \frac{80}{96} = \frac{5}{6}$$

[14] 4. Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 6 \\ 2 & 0 & 6 \end{bmatrix}$.

You are given that the characteristic equation of A is $\lambda(1-\lambda)(\lambda-7)=0$.

(a) Find the eigenvalues of the matrix A.

(b) For each eigenvalue, find a basis for the corresponding eigenspace.

(c) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

a) The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 7$.

b) $\lambda_1 = 0$: $(A-\lambda I)\vec{x} = 0$.

$$(A - 0I)\vec{x} = 0 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 6 & 0 \\ 2 & 0 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is free

$x_2 = -6x_3$

$x_1 = -3x_3$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ -6x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -6 \\ 1 \end{bmatrix}$$

$\lambda_2 = 1$:

$$(A - I)\vec{x} = 0 \Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 3 & 0 \\ 0 & 0 & 6 & 0 \\ 2 & 0 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda_3 = 7$:

$$(A - 7I)\vec{x} = 0 \Rightarrow \left[\begin{array}{ccc|c} -6 & 0 & 3 & 0 \\ 0 & -6 & 6 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is free

$x_2 = x_3$

$x_1 = \frac{1}{2}x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_3 \\ x_3 \\ x_3 \end{bmatrix} = \frac{1}{2}x_3 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$E_0 = \text{Span}\left\{\begin{bmatrix} -3 \\ -6 \\ 1 \end{bmatrix}\right\}, \quad E_1 = \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}, \quad E_7 = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\right\}$$

c) $$P = \begin{bmatrix} -3 & 0 & 1 \\ -6 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

[10] 5. Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix} \right\}$.

(a) Find a basis for W. What is the dimension of W?

(b) Write $x = \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix}$ as a linear combination of the basis vectors of W, which you found in part (a).

a) $$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 3 & 1 & 4 & 5 \\ -1 & 2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & -5 & -5 & -10 \\ 0 & 4 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow \ \uparrow$ pivot columns

So, a basis for W is

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}. \quad \dim W = 2$$

b) $$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & 1 & 7 \\ -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -5 & -5 \\ 0 & 4 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

So, $$\begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

[12] 6. Let $u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $W = \text{Span}\{u_1,\ u_2,\ u_3\}$ and $x = \begin{bmatrix} 6 \\ 4 \\ 8 \\ 10 \end{bmatrix}$.

(a) Show that $\{u_1,\ u_2,\ u_3\}$ is an orthogonal set.

(b) Find the orthogonal projection of the vector x onto W.

(c) Write x as the sum of a vector in W and a vector orthogonal to W.

(d) Find the distance from x to W.

a) $u_1 \cdot u_2 = 0$, $u_1 \cdot u_3 = 0$, $u_2 \cdot u_3 = 0$

b) $$\hat{x} = \text{proj}_W x = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{x \cdot u_3}{u_3 \cdot u_3} u_3$$

$$= \frac{6-4+8+10}{1+1+1+1} u_1 + \frac{6+12+8+10}{1+9+1+1} u_2 + \frac{-12+0+8+10}{4+0+1+1} u_3$$

$$= 5u_1 + 3u_2 + u_3 = \begin{bmatrix} 6 \\ 4 \\ 9 \\ 9 \end{bmatrix}$$

c) $$z = x - \hat{x} = \begin{bmatrix} 6 \\ 4 \\ 8 \\ 10 \end{bmatrix} - \begin{bmatrix} 6 \\ 4 \\ 9 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

d) $$\text{dist}(x, W) = \text{dist}(x, \hat{x}) = \| x - \hat{x} \|$$

$$= \| z \| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

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