MAT2322 Notes - By Eric Hua

Contents

Pre-knowledge

1. Trig Identities:

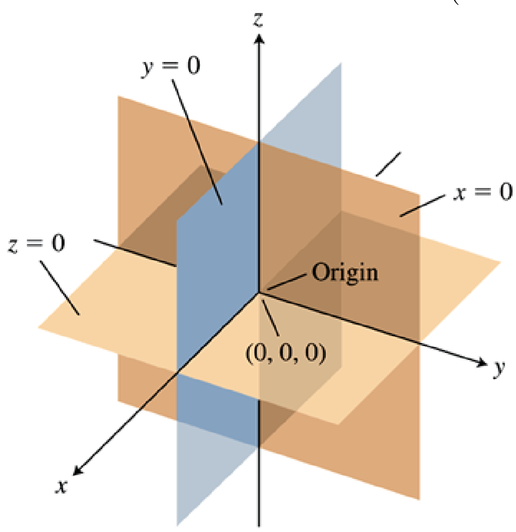
   - $\cos^2 x = \dfrac{1+\cos 2x}{2}$, $\quad \sin^2 x = \dfrac{1-\cos 2x}{2}$, $\quad \sin 2x = 2\sin x \cos x$.
   - $\sin a \sin b = \dfrac{\cos(a-b)-\cos(a+b)}{2}$, $\quad \cos a \cos b = \dfrac{\cos(a+b)+\cos(a-b)}{2}$,
   - $\sin a \cos b = \dfrac{\sin(a+b)+\sin(a-b)}{2}$.

2. Integration:

   - Fundamental Theorem of Calculus: $\displaystyle\int_a^b f(x)dx = F(b) - F(a)$, where $F'(x) = f(x)$.
   - Integration by parts: $\displaystyle\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$, or $\displaystyle\int udv = uv - \int vdu$.
   - Integration by substitution: $\displaystyle\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$, where $u = g(x)$.
   - Trig integral: $\displaystyle\int \sin^m x \cos^n x dx$:
     - If m is odd, then let $u = \cos x$.
     - If n is odd, then let $u = \sin x$.
     - If m and n are even, then use half-angle formula.
   - Trigonometric substitutions:
     - $\sqrt{a^2 - x^2}$: Let $x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$;
     - $\sqrt{a^2 + x^2}$: Let $x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$;
     - $\sqrt{x^2 - a^2}$: Let $x = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$.
   - Partial fractions.

12.1-12.5: Vectors and the Geometry of Space

1. The three-dimensional Cartesian (Rectangular) coordinate system (x, y, z), 8 octants:



- Coordinate Planes: $x = 0$ is yz-plane; $y = 0$ is xz-plane; $z = 0$ is xy-plane.

2. Vectors:

- Vectors in $\mathbb{R}^n$: $\vec{v}$, or, $\mathbf{v}= (v_1, ..., v_n) = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, zero vector $\vec{0} = (0, ..., 0)$.

- Standard basis vectors in $\mathbb{R}^2$: $\vec{i} = (1, 0), \vec{j} = (0, 1)$. Position vectors can be expressed in terms of standard basis vectors: $(a, b) = a\vec{i} + b\vec{j}$.

- Standard basis vectors in $\mathbb{R}^3$: $\vec{i} = (1, 0, 0), \vec{j} = (0, 1, 0), \vec{k} = (0, 0, 1)$. Position vectors can be expressed in terms of standard basis vectors: $(a, b, c) = a\vec{i} + b\vec{j} + c\vec{k}$.

- Length (norm, magnitude) $|(v_1, ..., v_n)| = \sqrt{v_1^2 + ... + v_n^2}$.

- Sum: Let $\vec{u} = (u_1, ..., u_n)$, $\vec{v} = (v_1, ..., v_n)$, then $\vec{u} + \vec{v} = (u_1 + v_1, ..., u_n + v_n)$.

- Scalar multiple: Let $\vec{u} = (u_1, ..., u_n)$, c be a scalar, then $c\vec{u} = (cu_1, ..., cu_n)$.

- unit vector: $|\vec{u}| = 1$. For any nonzero vector $\vec{v}$, $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$ is an unit vector.

- Dot product: Let $\vec{u} = (u_1, ..., u_n)$, $\vec{v} = (v_1, ..., v_n)$, then $\vec{u} \cdot \vec{v} = u_1v_1 + ... + u_nv_n$.

- Cross product: Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, then

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, +u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

3 Line:

A line is determined by a point P and a nonzero vector (direction vector) $\vec{v}$ which is parallel to the line.

- Point-parallel form (vector form): $\vec{r}(t) = P + t\vec{v}$, $t \in \mathbb{R}$, $\vec{r} = (x, y, z)$ in 3D, and $\vec{v} = (x, y)$ in 2D.

- Parametric form: $x = p_1 + tv_1$, $y = p_2 + tv_2$, $z = p_3 + tv_3$.

4. Plane: A plane Π is determined by a point and a normal vector $\vec{n} = (a, b, c)$ which is perpendicular to the plane. Standard equation of a plane is:

$$ax + by + cz = d.$$

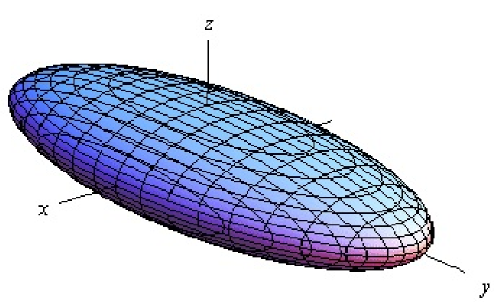
12.6 Cylinders and Quadric Surfaces

A **quadric surface** is the graph of a second degree equation in three variables x, y, z:

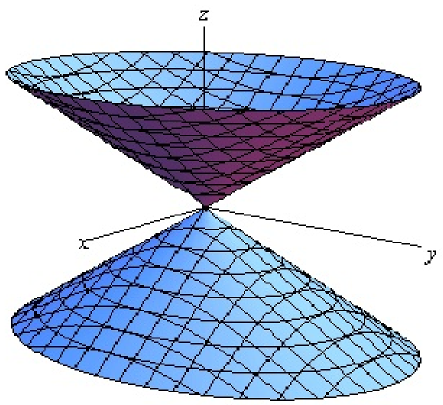
$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0.$$

By rotating the surface, or equivalently, rotating the axes, we may assume that $D = E = F = 0$.

1. Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $a, b, c > 0$.



2. Cone: The general equation of a cone is: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$, $a, b, c > 0$.



3. Cylinder

A cylinder is a surface consisting of lines parallel (rulings) to a given line passing through a given plane curve. In most cases, the rulings are parallel to an axis. In this case, the equation of the surface contains only two variables, which gives the plane curve in a coordinate plane. For example,

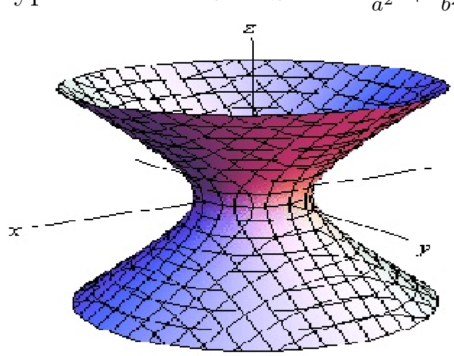
(i) $x^2 + y^2 = 1$. This is a cylinder containing lines parallel to the z-axis passing through points on the unit circle in the xy- plane.

(ii) $z = x^2$. This is a cylinder containing lines parallel to the y-axis passing through points on the curve $z = x^2$ in the xz- plane.

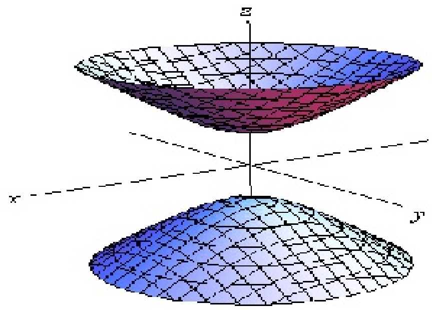
(iii) $yz = 1$. This is a cylinder containing lines parallel to the x-axis passing through points on two branches of the curve $yz = 1$ in the yz- plane.

(iv) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is a cylinder containing lines parallel to the z-axis passing through points on the ellipse in the xy- plane.

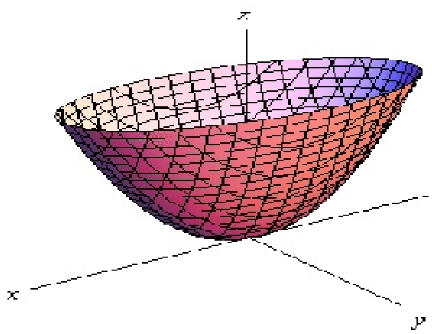
4. Hyperboloid of One Sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, $a, b, c > 0$.



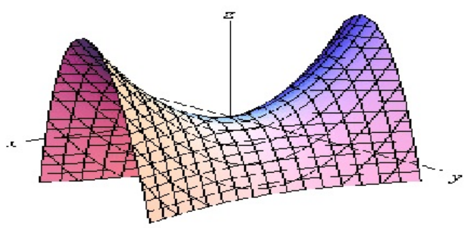
5. Hyperboloid of Two Sheets: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $a, b, c > 0$.



6. Elliptic Paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$, $a, b, c > 0$.



7. Hyperbolic Paraboloid: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$, $a, b, c > 0$.



Example 1. *Classify the following quadric surfaces:*

(1) $z = -x^2 - y^2 + 4$. This is an elliptic paraboloid that opens downward.

(2) $4x^2 - y^2 + z^2 - 8x - 2y - 6z - 4 = 0$. This is Hyperboloid of One Sheet with centre (1,-1,3), and centered on the line which is parallel to z-axis.

(3) $4x^2 - y^2 + 2z^2 + 4 = 0$. This is a hyperboloid of two sheets centered at the x-axis.

(4) $x^2 + 2z^2 - 6x - y + 10 = 0$. This is an elliptic paraboloid with vertex (3, 1, 0), centered with line x = 3, y = 1.

Remark. If a quadratic equation of x, y, and z with terms xy, yz, or xz, then a rotation is needed to convert it to the standard form.

Chapter 13: Vector Functions

13.1 Vector functions and Space Curves

A **vector function (or vector-valued function, or curve C)** is a function that takes one or more variables and returns a vector.

- 2D vector functions: $\vec{r}(t) = (x, y) = (f(t), g(t)) = f(t)\vec{i} + g(t)\vec{j}$.
- 3D vector functions: $\vec{r}(t) = (x, y, z) = (f(t), g(t), h(t)) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$,

where $x = f(t), y = g(t), z = h(t)$ are called component functions, or parametric equations of the curve C.

Example 2. *The vector function* $\vec{r}(t) = (3 - 2t, 2, 3 - 5t) = (3, 2, 3) + t(-2, 0, -5)$ *is a line.*

Example 3. *Find a vector equation that represents the intersection between a cylinder* $\frac{x^2}{2^2} + \frac{y^2}{4^2} = 1$ *and the plane* $y - z = 3$.

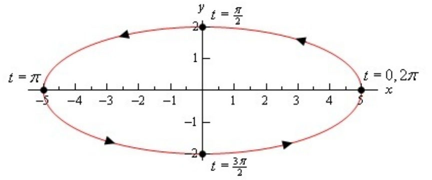
Solution: The intersection is a curve C, whose projection onto xy-plane is an ellipse. Let $x = 2\cos t, y = 4\sin t$, $0 \le t \le 2\pi$. Then $z = y - 3 = 4\sin t - 3$. Thus

$$\vec{r}(t) = 2\cos t\vec{i} + 4\sin t\vec{j} + (4\sin t - 3)\vec{k}, \quad 0 \le t \le 2\pi,$$

which is called parametrization of C.

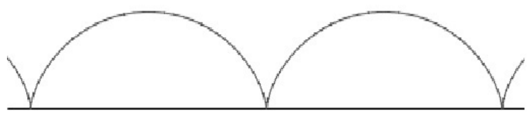
Example 4. *Sketch the vector function:*

$$\vec{r}(t) = (5\cos t, \sin t), \quad 0 \le t \le 2\pi.$$



Example 5. *Sketch the Cycloid* $\vec{r}(t) = (x, y)$, *where*

$$x = r(t - \sin t), y = r(1 - \cos t), , -\infty < t < \infty.$$



13.2 Derivatives and Integrals of Vector functions

- Derivative (Tangent vector): $\vec{r}\,'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k} = (x'(t), y'(t), z'(t))$.
- The tangent line to the curve at a point t_0 is the line with $\vec{r}\,'(t_0)$ as the direction vector.
- Normal plane to a curve $\vec{r}(t) = (x(t), y(t), z(t))$ at the point $P(a, b, c)$, where $a = x(t_0)$, $b = y(t_0)$, $c = z(t_0)$, is:

$$x'(t_0)(x-a) + y'(t_0)(y-b) + z'(t_0)(z-c) = 0.$$

Example 6. *Let* $C: \ \vec{r}(t) = (t, 2\sin t, 2\cos t), \quad 0 \le t \le \pi$. *Find the tangent line and normal plane to the curve at* $t = \pi/3$.

Solution: (a) Tangent line:

$$\vec{r}\,'(t) = (1, 2\cos t, -2\sin t).$$

$$\vec{r}(\pi/3) = (\pi/3, \sqrt{3}, 1), \quad \vec{r}\,'(\pi/3) = (1, 1, -\sqrt{3}).$$

The tangent line is:

$$(x, y, z) = (\pi/3, \sqrt{3}, 1) + t(1, 1, -\sqrt{3}).$$

(b) The normal plane is

$$(1, 1, -\sqrt{3}) \cdot (x - \pi/3, y - \sqrt{3}, z - 1) = 0, i.e., x + y - \sqrt{3}z - \pi/3 = 0.$$

Integrals:

Antiderivative:

$$\int \vec{r}(t)dt = (\int f(t)dt, \int g(t)dt, \int h(t)dt) = \left(\int f(t)dt\right)\vec{i} + \left(\int g(t)dt\right)\vec{j} + \left(\int h(t)dt\right)\vec{k}.$$

Definite Integral:

$$\int_a^b \vec{r}(t)dt = (\int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt) = \left(\int_a^b f(t)dt\right)\vec{i} + \left(\int_a^b g(t)dt\right)\vec{j} + \left(\int_a^b h(t)dt\right)\vec{k}.$$

Example 7. *Let $\vec{r}(t) = (2\cos t, \sin t, 2t)$. Then*

$$\int \vec{r}(t)dt = (\int f(t)dt, \int g(t)dt, \int h(t)dt) = \left(\int f(t)dt\right)\vec{i} + \left(\int g(t)dt\right)\vec{j} + \left(\int h(t)dt\right)\vec{k}$$

$$= (2\sin t)\vec{i} - (\cos t)\vec{j} + (t^2)\vec{k} + \vec{c},$$

$$\int_0^{\pi/2} \vec{r}(t)dt = 2\vec{i} + \vec{j} + \frac{\pi^2}{4}\vec{k}.$$

13.3 Arc length and Curvature

Arc length of the parametric curve $\vec{r}(t)$, $\alpha \le t \le \beta$: $L = \int_\alpha^\beta |\vec{r}\,'(t)|dt.$

- If $\vec{r}(t) = (x(t), y(t))$, then $L = \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$
- If $\vec{r}(t) = (x(t), y(t), z(t))$, then $L = \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}dt.$

Example 8. *Find the length under one arch of cycloid:*

$$x = r(t - \sin t), y = r(1 - \cos t),\ 0 \le t \le 2\pi.$$

Solution:

$$L = \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2}dt = \int_0^{2\pi} \sqrt{[r(1-\cos t)]^2 + [r\sin t]^2}dt$$

$$= r\int_0^{2\pi} \sqrt{2 - 2\cos t}dt = r\int_0^{2\pi} 2\left|\sin\frac{t}{2}\right| dt = 8r.$$

Example 9. *Find the length of the curve:*

$$C:\ x = e^t + e^{-t}, y = 5 - 2t,\ 0 \le t \le 3.$$

Solution:

$$L = \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2}dt = \int_0^3 \sqrt{(e^t - e^{-t})^2 + (-2)^2}dt$$

$$= \int_0^3 \sqrt{(e^t + e^{-t})^2}dt = \int_0^3 (e^t + e^{-t})dt = e^3 - e^{-3}.$$

Example 10. *Find the length of the curve* $\vec{r}(t) = 8t\vec{i} + 3\sin(2t)\vec{j} + 3\cos(2t)\vec{k}, 0 \le t \le 2\pi.$

Solution:

$$\vec{r}\,'(t) = 8\vec{i} + 6\cos(2t)\vec{j} - 6\sin(2t)\vec{k}, \Rightarrow |\vec{r}\,'(t)| = 10,$$

$$L = \int_0^{2\pi} |\vec{r}\,'(t)|dt = \int_0^{2\pi} 10dt = 20\pi.$$

Example 11. *Find the length of the curve* C*, where* C *is the cycloid* $x = t, y = t^2, z = \frac{2}{3}t^3$*,* $0 \le t \le 1$.

Solution:

$$L = \int_0^1 |\vec{r}\,'(t)|dt = \int_0^1 \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}dt$$

$$= \int_0^1 \sqrt{1 + 4t^2 + 4t^4}dt = \int_0^1 \sqrt{(1 + 2t^2)^2}dt = \int_0^1 (1 + 2t^2)dt = \frac{5}{3}.$$

The Arc Length Function:

We call

$$s = s(t) = \int_a^t |\vec{r}\,'(u)|du$$

the arc length function. We have

$$s'(t) = |\vec{r}\,'(t)|.$$

Example 12. *Given the curve* $\vec{r}(t) = 8t\vec{i} + 3\sin(2t)\vec{j} + 3\cos(2t)\vec{k}.$

(1) Find the arc length function.
(2) Reparametrize the curve $\vec{r}(t)$ with respect to arc length measured from (0,0,3).
(3) Where on the curve are we after travelling for a distance of 5π?

Solution: (1).

$$s = s(t) = \int_0^t |\vec{r}\,'(u)|du = 10t.$$

(2). $s = 10t$, so $t = 0.1s$,

$$\vec{r}(s) = \vec{r}(t(s)) = 0.8s\vec{i} + 3\sin(0.2s)\vec{j} + 3\cos(0.2s)\vec{k}.$$

(3). $s = 5\pi$,

$$\vec{r}(s) = \vec{r}(t(s)) = 0.8s\vec{i} + 3\sin(0.2s)\vec{j} + 3\cos(0.2s)\vec{k} = 4\pi\vec{i} + 3\sin(\pi)\vec{j} + 3\cos(\pi)\vec{k}.$$

So we are at $(4\pi, 0, -3)$.
Curvature:

The curvature measures how fast a curve C is changing direction at a given point P. The curvature of a curve $\vec{r}(t)$ is:

$$\kappa(t) = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}.$$

The radius of curvature of C at P is

$$\rho = \frac{1}{\kappa}.$$

Example 13. *The curvature of a circle of radius r is 1 / r.*

Example 14. *Let $\vec{r}(t) = (t, t^2, t^3)$. Find $\kappa(t)$.*

Solution: $\vec{r}\,'(t) = (1, 2t, 3t^2), \vec{r}\,''(t) = (0, 2, 6t) \Rightarrow$

$$|\vec{r}\,' \times \vec{r}\,''| = |(6t^2, -6t, 2)| = 2\sqrt{9t^4 + 9t^2 + 1}, \Rightarrow$$

$$\kappa(t) = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{\sqrt{9t^4 + 4t^2 + 1}^3}.$$

Example 15. *Let $\vec{r}(t) = (\cos t, \sin t, t)$. Then $\kappa(t) = \frac{1}{2}$.*

Special case: if the curve is in the xy- plane, with $y = f(x)$, then

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.$$

Example 16. *Find x where the graph of $y = e^x$ has the maximum curvature.*

Solution:

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{e^x}{[1 + e^{2x}]^{3/2}}, \Rightarrow$$

$$\kappa'(x) = \frac{e^x(1 + e^{2x})^{1/2}(1 - 2e^{2x})}{[1 + e^{2x}]^3} = 0 \Rightarrow x = -\frac{\ln 2}{2}.$$

Chapter 14 Partial Derivatives

14.1 Functions

- A function of two variables $z = f(x, y)$ is a rule which maps each point (x, y) in a set D to a unique number z. The set D is called the domain of the function, which is often denoted $D(f)$. Level curves (contour maps) of $f(x, y)$: $f(x, y) = k$ for different k.

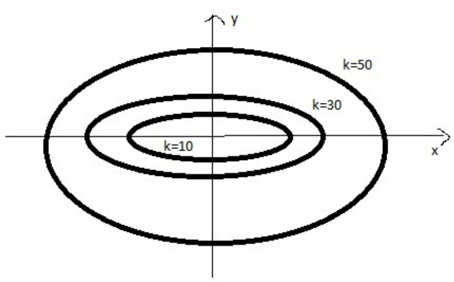
- A function of three variables $w = f(x, y, z)$ is a rule which maps each point (x, y, z) in a set D to a unique number w. The set D is called the domain of the function, which is often denoted $D(f)$. Level surfaces of $f(x, y, z)$: $f(x, y, z) = k$ for different k.

Example 17. *Sketch three level curves to the function $f(x, y) = 2x^2 + 5y^2$.*

Solution: When $z = k = 50$: $2x^2 + 5y^2 = 50$, $\dfrac{x^2}{25} + \dfrac{y^2}{10} = 1$.

When $z = k = 20$: $2x^2 + 5y^2 = 20$, $\dfrac{x^2}{10} + \dfrac{y^2}{4} = 1$.

When $z = k = 10$: $2x^2 + 5y^2 = 10$, $\dfrac{x^2}{5} + \dfrac{y^2}{1} = 1$.



14.2 Limits and continuity

- If $f(x, y)$ can be made as close to L as (x, y) close to (a, b), then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L.$$

In general, if there are different limits when (x, y) approaches (a, b) along different paths, then the limit does not exist. $f(x, y)$ is continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

- If $f(x, y, z)$ can be made as close to L as (x, y, z) close to (a, b, c), then

$$\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) = L.$$

If $L = f(a, b, c)$, then f is continuous at (a, b, c).

Example 18. *Assume that the limit* $\lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^2+y^2}$ *exists. Find it.*

Solution: By taking the path $y = x$, then $\frac{4x^2y}{x^2+y^2} = 2x \to 0$.

Example 19. *Show that the limit* $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ *does not exist.*

Solution: Since the limit is $1/2$ when $y = x$; and the limit is 0 when $y = 0$.

Example 20.

$$\lim_{(x,y,z)\to(1,2,3)} \frac{xy + yz + xz}{xyz - 1} = \frac{11}{5}.$$

14.3 Partial Derivatives

- Partial derivatives of $z = f(x, y)$:

$$z_x = \frac{\partial z}{\partial x} := \frac{\partial f}{\partial x} := f_x(x, y) := D_x f := \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

which is the derivative of f with respect to x;

$$z_y = \frac{\partial z}{\partial y} := \frac{\partial f}{\partial y} := f_y(x, y) := D_y f := \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h},$$

which is the derivative of f with respect to y.

  - Methods:
    1. To find f_x: regard y as a constant, and differentiate $f(x, y)$ with respect to x;
    2. To find f_y: regard x as a constant, and differentiate $f(x, y)$ with respect to y.
  - Meaning: f_x means the rate of change of f with respect to x when y is fixed.

- Partial derivatives of $w = f(x, y, z)$:

$$\frac{\partial w}{\partial x} := \frac{\partial f}{\partial x} := f_x(x, y, z) := D_x f := \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h},$$

  which is the derivative of f with respect to x.

  - Meaning: f_x means the rate of change of f (or w) with respect to x when y and z are fixed.

Example 21. *Let*

$$f(x) = \begin{cases} \frac{x^3+y}{x^2+y^2} & if\ (x,y) \neq (0,0) \\ 0 & if\ (x,y) = (0,0) \end{cases}$$

(a) Using the definition of a partial derivative (do not differentiate) find $f_x(0,0)$.

(b) Using the definition of a partial derivative (do not differentiate) find $f_y(0,0)$.

Solution:
(a)

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1.$$

(b)

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{\frac{1}{k} - 0}{k} \nexists.$$

Remark. This example shows that the existence of partial derivatives at a point is insufficient to guarantee that the function is continuous there.

Example 22. *Let $f(x, y, z) = (\sin z)e^{xy} \ln x$. Calculate f_x, f_y, f_z.*

Solution:

$$f_x = (\sin z)ye^{xy} \ln x + (\sin z)e^{xy}(\frac{1}{x}), \quad f_y = (\sin z)xe^{xy} \ln x, \quad f_z = (\cos z)e^{xy} \ln x.$$

Higher derivatives:

f_{xx}, $\frac{\partial^3 f}{\partial z \partial y \partial x} = f_{xyz}$,...

Example 23. *Let* $f(x, y, z) = \sin ze^{xy} \ln x$. *Calculate* f_{xyz}.

Solution: $f_x(x, y, z) = y \sin ze^{xy} \ln x + \sin ze^{xy}/x$.
$f_{xy}(x, y, z) = \sin ze^{xy} \ln x + xy \sin ze^{xy} \ln x + \sin ze^{xy}$.
$f_{xyz}(x, y, z) = \cos ze^{xy} \ln x + xy \cos ze^{xy} \ln x + \cos ze^{xy}$.

14.4 Tangent Planes

Surfaces in 3D: The surface S given by $F(x, y, z) = 0$.

- The tangent plane of the surface S given by $F(x, y, z) = 0$ at $P(a, b, c)$ is the plane that passes through P and has normal vector $\vec{n} = (F_x(a, b, c), F_y(a, b, c), F_z(a, b, c))$. Thus the equation is

$$(F_x(a, b, c), F_y(a, b, c), F_z(a, b, c)) \cdot (x - a, y - b, z - c) = 0.$$

- The normal line to the surface S given by $F(x, y, z) = 0$ at $P(a, b, c)$ is the line that passes through P and has the direction vector $\vec{n}$. The equation is:

$$(x, y, z) = (a, b, c) + t(F_x(a, b, c), F_y(a, b, c), F_z(a, b, c)), \quad t \in \mathbb{R}.$$

- Equation of the tangent plane of $z = f(x, y)$ at (x_0, y_0):

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Example 24. *Find the equation of the tangent plane of the surface* $z = e^{x+y} - \frac{x}{y}$ *at the point* $(1, -1, 2)$.

Solution: Let $f(x, y) = e^{x+y} - \frac{x}{y}$. Then

$$f_x = e^{x+y} - \frac{1}{y}, \quad f_x(1, -1) = 2.$$

$$f_y = e^{x+y} + \frac{x}{y^2}, \quad f_y(1, -1) = 2.$$

Thus the equation of the tangent plane at the point $(1, -1, 2)$ is

$$z - 2 = 2(x - 1) + 2(y + 1), \quad \textit{i.e.,} \quad 2x + 2y - z + 2 = 0.$$

Example 25. *Find the equation of the tangent plane and the normal line at the point (2,1,9) to the ellipsoid* $\frac{x^2}{12} + \frac{y^2}{3} + \frac{z^2}{27} = \frac{11}{3}$.

Solution: Let $F(x, y, z) = \frac{x^2}{12} + \frac{y^2}{3} + \frac{z^2}{27} - \frac{11}{3}$. Then the normal vector of the tangent plane is

$$\vec{n} = (F_x(2, 1, 9), F_y(2, 1, 9), F_z(2, 1, 9)) = (1/3, 2/3, 2/3).$$

Tangent plane: $x + 2y + 2z - 22 = 0$.

The normal line is: $(x, y, z) = (2, 1, 9) + t(1/3, 2/3, 2/3)$.

14.5 The Chain Rule

- Basic Chain Rule

  1. If $z = f(x, y)$, $x = g(t)$, $y = h(t)$, then
  $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$
  2. If $w = f(x, y, z)$, $x = g(t)$, $y = h(t)$, $z = k(t)$, then
  $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

- General Chain Rule:

  1. If $z = f(x, y)$, $x = g(u, v)$, $y = h(u, v)$, then
  $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u},$$
  $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$
  2. If $w = f(x, y, z)$, $x = g(u, v)$, $y = h(u, v)$, $z = k(u, v)$, then
  $$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u},$$
  $$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

- Implicit Differentiation:

1. If $F(x,y)=0$, then
$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$
Here when we calculate partial derivatives, we consider x and y as independent variables.

2. If $F(x,y,z)=0$, then
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$
Here when we calculate partial derivatives, we consider x, y and z as independent variables.

Example 26. *Find* $\frac{\partial z}{\partial x}$ *and* $\frac{\partial z}{\partial y}$ *if* $x^2y^3 + z^4 + 5xyz = 3$.

Solution: Let $F(x,y,z) = x^2y^3 + z^4 + 5xyz - 3$. Then

$$F_x = 2xy^3 + 5yz, F_z = 4z^3 + 5xy$$

Thus
$$z_x = -\frac{F_x}{F_z} = -\frac{2xy^3 + 5yz}{4z^3 + 5xy}.$$

14.6 Directional Derivatives and the Gradient Vector

- The directional derivative of the function $f(x,y)$ at (x_0,y_0) in the direction of a unit vector $\vec{u} = (u_1,u_2)$ is

$$\begin{aligned} f_{\vec{u}}(x_0,y_0) \text{ or } D_{\vec{u}}f(x_0,y_0) &= \lim_{h\to 0} \frac{f(x_0+hu_1, y_0+hu_2) - f(x_0,y_0)}{h} \\ &= f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2 \\ &= (f_x(x_0,y_0), f_y(x_0,y_0)) \cdot (u_1,u_2). \end{aligned}$$

  $f_{\vec{u}}(x_0,y_0)$ means the rate of change of $f(x,y)$ at (x_0,y_0) in the direction of $\vec{u}$.

- The gradient of $f(x,y)$ at (x_0,y_0) is

$$\nabla f(x_0,y_0) \text{ or } \mathrm{grad} f(x_0,y_0) = (f_x(x_0,y_0), f_y(x_0,y_0)).$$

  1. $\nabla f(x_0,y_0)$ points into the direction of maximum increase of f at (x_0,y_0).
  2. $\nabla f(x_0,y_0)$ is perpendicular to the contour line (or level curve) of f through (x_0,y_0).

3. $|\nabla f(x_0, y_0)|$ is the maximum rate of change of f at (x_0, y_0).

- The directional derivative of the function $f(x, y, z)$ at (x_0, y_0, z_0) in the direction of a unit vector $\vec{u} = (u_1, u_2, u_3)$ is

$$f_{\vec{u}}(x_0, y_0, z_0) = D_{\vec{u}} f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)u_1 + f_y(x_0, y_0, z_0)u_2 + f_z(x_0, y_0, z_0)u_3.$$

- The gradient of $f(x, y, z)$ at (x_0, y_0, z_0) is

$$\text{grad} f(x_0, y_0, z_0) \textit{ or } \nabla f(x_0, y_0, z_0) = (f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0)).$$

1. $\nabla f(x_0, y_0, z_0)$ points into the direction of maximum increase of f at (x_0, y_0, z_0).
2. $\nabla f(x_0, y_0, z_0)$ is perpendicular to the level surface of f through (x_0, y_0, z_0).
3. $|\nabla f(x_0, y_0, z_0)|$ is the maximum rate of change of f at (x_0, y_0, z_0).

Example 27. *Suppose that the temperature of a room at a point (x, y, z) is given by*

$$T(x, y, z,) = \frac{80}{1 + x^2 + 2y^2 + 3z^2}\ C^o.$$

(1) In which direction does the temperature increase fastest at the point $(2, 1, 1)$?
(2) What is the maximum rate of increase?
(3) Find the directional derivative of $T(x, y, z)$ at the point $(2, 1, 1)$ in the direction of the vector $\vec{v} = (1, -2, -2)$.

Solution: (1)

$$\nabla T(x, y, z) = (T_x, T_y, T_z) = \frac{160}{(1 + x^2 + 2y^2 + 3z^2)^2}(-x, -2y, -3z),$$

$$\nabla T(2, 1, 1) = \frac{8}{5}(-2, -2, -3).$$

(2)

$$|\nabla T(2, 1, 1)| = \frac{8\sqrt{17}}{5}.$$

(3) The direction of the unit vector is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = (\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}).$$

$$T_{\vec{u}}(2, 1, 2) = \nabla T(2, 1, 1) \cdot \vec{u} = \frac{8}{5}(-2, -2, -3) \cdot (\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}) = \frac{64}{15}.$$

14.7 Maximum and minimum values

Part 1. Local extrema.

Definition 1. *We say that a function* $f(x, y)$ *has a relative (local) minimum at a point* (x_0, y_0) *if there is a circle centered at* (x_0, y_0) *such that*

$$f(x, y) \geq f(x_0, y_0)$$

for all (x, y) *in that circle;* $f(x, y)$ *has a relative (local) maximum at a point* (x_0, y_0) *if there is a circle centered at* (x_0, y_0) *such that*

$$f(x, y) \leq f(x_0, y_0)$$

for all (x, y) *in that circle.*

We say that $f(x, y)$ *has the absolute (global) minimum at a point* (x_0, y_0) *if*

$$f(x, y) \geq f(x_0, y_0)$$

for all (x, y) *in the domain;* $f(x, y)$ *has the absolute (global) maximum at a point* (x_0, y_0) *if*

$$f(x, y) \leq f(x_0, y_0)$$

for all (x, y) *in the domain.*

Definition 2. *The critical points of a function* $f(x, y)$ *are those points* (x_0, y_0) *for which* $f_x(x_0, y_0) = 0$ *and* $f_y(x_0, y_0) = 0$*, or if* $f_x(x_0, y_0)$ *or* $f_y(x_0, y_0)$ *is undefined. Saddle point: The graph of the function crosses the tangent plane at this point.*

First-Partials Test for Relative Extrema: If f has a relative extrema at (a, b), and the first partial derivatives exist in a circle centered at (a, b), then (a, b) is a critical point.

Second-Partials Test for Relative Extrema: Assume that f has a continuous partial derivatives on an open region containing (a, b). Let (a, b) be a critical point of f. Denote

$$d(a, b) = f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2 .$$

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a relative minimum.

2. If $d > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a relative maximum.

3. If $d < 0$, then (a, b) is a saddle point.

4. If $d = 0$, then the second derivatives test gives nothing.

Example 28. *Classify the critical points of* $f(x, y) = x^3 - 3xy + y^3$

Solution:

$$f_x(x, y) = 3x^2 - 3y, \quad f_y(x, y) = -3x + 3y^2, \Rightarrow$$

$$f_{xx}(x, y) = 6x, \quad f_{yy}(x, y) = 6y, \quad f_{xy}(x, y) = -3.$$

Setting $f_x = 0$ and $f_y = 0 :\ 3x^2 - 3y = 0, -3x + 3y^2 = 0.$ We imply that $(x, y) = (0, 0), (1, 1)$.

$$d(x, y) = f_{xx}(x, y) f_{yy}(x, y) - (f_{xy}(x, y))^2 = 36xy - 9.$$

- $d(0, 0) = -9$, so $(0, 0)$ is a saddle point.
- $d(1, 1) = 27,\ f_{xx}(1, 1) = 6 > 0$, so $f(1, 1)$ is a relative minimum.

Example 29. *Find and classify the critical points of the function* $f(x, y) = 12x^2y + y^3 - 24x^2 - 6y^2$.

Solution:

$$f_x = 24xy - 48x, \quad f_y = 12x^2 + 3y^2 - 12y,$$

$$f_{xx} = 24y - 48 \qquad f_{yy} = 6y - 12 \qquad f_{xy} = 24x$$

$$f_x = 0$$
$$f_y = 0$$

$$24xy - 48x = 0$$
$$12x^2 + 3y^2 - 12y = 0$$

$$24x\,(y - 2) = 0 \quad \Rightarrow x = 0 \ \ or \ \ y = 2$$

If $x = 0$, then

$$3y^2 - 12y = 0 \qquad \Rightarrow \qquad y = 0,\ y = 4$$

If $y = 2$, then

$$12x^2 + 3y^2 - 12y = 12\left(x^2 - 1\right) = 0 \qquad \Rightarrow \qquad x = -1,\ x = 1$$

Critical points are: (0,0), (0,4), (1,2), (-1,2).

$$d(x,y) = (24y-48)(6y-12) - (24x)^2 = 144\left[(y-2)^2 - 4x^2\right]$$

$$d(0,0) > 0 \qquad f_{xx}(0,0) < 0$$
$$d(0,4) > 0 \qquad f_{xx}(0,4) > 0$$
$$d(-1,1) < 0$$
$$d(1,1) < 0$$

$(0,0)$	:	Relative Maximum
$(0,2)$	:	Relative Minimum
$(1,1)$	:	Saddle Point
$(-1,1)$	:	Saddle Point

Example 30. *Find the critical point(s) of* $f(x,y) = x(x-2)y(y+4)$ *and classify them.*

Solution:

$$f_x = (2x-2)y(y+4), \quad f_y = x(x-2)(2y+4).$$

Set

$$f_x = 0, \quad f_y = 0,$$

i.e.,

$$(2x-2)y(y+4) = 0, \quad x(x-2)(2y+4) = 0.$$

So critical points are

$$(1,-2), (0,0), (0,-4), (2,0), (2,-4).$$

To test all of them, we use the Second-Partials Test.

$$f_{xx}(x,y) = 2y(y+4), \quad f_{yy}(x,y) = 2x(x-2), \quad f_{xy}(x,y) = (2x-2)(2y+4).$$

$$d(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2.$$

- $d(1,-2) = 16 > 0$, $f_{xx}(1,-2) = -2 < 0$, so $f(1,-2)$ is a relative max.
- $d(0,0) = -64 < 0$, so (0,0) is a saddle point.
- $d(0,-4) = -64 < 0$, so (0,-4) is a saddle point.
- $d(2,0) = -64 < 0$, so (2,0) is a saddle point.

- $d(2,-4) = -64 < 0$, so (2,-4) is a saddle point.

Example 31. *Find the shortest distance from the point* $(0,0,-2)$ *to the plane* $\Pi: \ x + 2y + z = 4$.

Solution: The distance from any point $(x,y,z) \in \Pi$ to the point $(0,0,-2)$ is

$$d = \sqrt{x^2 + y^2 + (z+2)^2}, \ z = 4 - x - 2y.$$

$$d = \sqrt{x^2 + y^2 + (6 - x - 2y)^2}.$$

Let

$$f(x,y) = x^2 + y^2 + (6 - x - 2y)^2.$$

When f has minimum, d will be minimum.

$$f_x = 2x - 2(6 - x - 2y), \ f_y = 2y - 4(6 - x - 2y).$$

From $f_x = 0, f_y = 0$ we imply that $(x,y) = (1,2)$, which is the only critical point.

$$f_{xx}(x,y) = 4, f_{yy}(x,y) = 10, f_{xy}(x,y) = 4.$$

$$D(1,2) = f_{xx}(1,2) f_{yy}(1,2) - f_{xy}(1,2)^2 = 24 > 0, \quad f_{xx}(1,2) > 0.$$

Thus we have local min at $(1,2)$, which should be global min.

$$d = \sqrt{6}.$$

Part 2. Finding the absolute extrema of the function $f(x)$ defined on closed region D.

- Find the critical points inside D, then values of $f(x)$ at the critical points.
- Find the max and min values of $f(x)$ on the boundary.
- Take the largest and smallest values of these extrema to get absolute max and min.

Example 32. *Find the absolute extrema of* $f(x,y) = x^3 - 3x + 9y^2 + 1$ *on the region* $D: \ \{(x,y) | -2 \le x \le 3, -1 \le y \le 4\}$.

Solution:

Step 1. Inside D.

$$f_x(x,y) = 3x^2 - 3, \quad f_y(x,y) = 18y.$$

Setting $f_x = 0$ and $f_y = 0$: We imply that $(x,y) = (1,0), (-1,0)$. They are critical points inside the rectangle.

$$f(1,0) = -1, \ f(-1,0) = 3.$$

Step 2. On the boundary of D.

Line segments L_1: $-2 \leq x \leq 3$, $y = -1$, let

$$g(x) = f(x,-1) = x^3 - 3x + 10.$$

$g'(x) = 0, \ x = \pm 1$. Note that

$$g(1) = 8, \ g(-1) = 12, \ g(-2) = 8, \ g(3) = 28.$$

Line segments L_2: $-2 \leq x \leq 3$, $y = 4$, let

$$g(x) = f(x,4) = x^3 - 3x + 145.$$

$g'(x) = 0, \ x = \pm 1$.

$$g(1) = 143, \ g(-1) = 147, \ g(-2) = 143, \ g(3) = 163.$$

Line segments L_3: $-1 \leq y \leq 4$, $x = -2$, let

$$g(y) = f(-2,y) = 9y^2 - 1.$$

$g'(y) = 0, \ y = 0$.

$$g(0) = -1, \ g(-1) = 8, \ g(4) = 143.$$

Line segments L_4: $-1 \leq y \leq 4$, $x = 3$, let

$$g(y) = f(3,y) = 9y^2 + 19.$$

$g'(y) = 0, \ y = 0$.

$$g(0) = 19, \ g(-1) = 28, \ g(4) = 163.$$

Therefore On the boundary of D: the minimum is $f(-2,0) = -1$, the maximum is $f(3,4) = 163$.

Step 3. Comparing values of f in Step 1 and Step 2, the absolute max $= 163$ at $(3,4)$, absolute min $= -1$, at $(-2,0)$ or $(1,0)$.

Example 33. *Find the absolute extrema of* $f(x, y) = x^3 - 3x + 9y^2 + 1$ *on the region* $4x^2 + 9y^2 \leq 36$.

Solution:

Step 1. Inside D**.**

$$f_x(x, y) = 3x^2 - 3, \quad f_y(x, y) = 18y.$$

Setting $f_x = 0$ and $f_y = 0:$ We imply that $(x, y) = (1, 0), (-1, 0)$. They are critical points inside the ellipse.

$$f(1, 0) = -1, \; f(-1, 0) = 3.$$

Step 2. On the boundary of D. $4x^2 + 9y^2 = 36$.

Method 1. Eliminating one variable:

$$9y^2 = 36 - 4x^2.$$

From $4x^2 \leq 36$ we imply that $-3 \leq x \leq 3$.

$$f(x, y) = x^3 - 3x + 9y^2 + 1 = x^3 - 3x + 37 - 4x^2 = g(x), \quad -3 \leq x \leq 3.$$

$$g'(x) = 0, \Rightarrow 3x^2 - 8x - 3 = 0, \Rightarrow x = -\frac{1}{3}, 3.$$

$$g(-\frac{1}{3}) = \frac{1013}{27}, \quad g(3) = 19, \quad g(-3) = -17.$$

Method 2: Let

$$x = 3\cos t, y = 2\sin t.$$

Then

$$g(t) = f(x, y) = 27\cos^3 t - 9\cos t + 36\sin^2 t + 1.$$

$$g'(t) = 81\cos^2 t(-\sin t) + 9\sin t + 72\sin t\cos t = -9\sin t(9\cos t + 1)(\cos t - 1).$$

$$g'(t) = 0, \quad \sin t = 0, \; \textit{or} \; \cos t = -1/9, \; \textit{or} \; \cos t = 1.$$

$$t = 0, \pi, 2\pi, \; \textit{or} \; \cos t = -1/9.$$

Critical points are $(3, 0)$, $(-3, 0)$, $(-\frac{1}{3}, \pm\frac{8\sqrt{5}}{9})$.

$$f(3, 0) = 19, \; f(-3, 0) = -17, \; f(-\frac{1}{3}, \pm\frac{8\sqrt{5}}{9}) = \frac{1013}{27}.$$

Step 3. Comparing values of f in Step 1 and Step 2, the absolute absolute max= $\frac{1013}{27}$ at $(-\frac{1}{3}, \pm\frac{8\sqrt{5}}{9})$, absolute min = -17, at $(-3, 0)$.

14.8 Lagrange Multipliers

Case 1. Two variables with one constraint: Find the extreme values of the function $z = f(x, y)$ subject to constraint $g(x, y) = 0$.

Interpretation. Find extreme values of $f(x, y)$ on a curve.

When z attains an extreme value a, the level curve $f(x, y) = a$ and $g(x, y) = 0$ have the same tangent line. Hence their gradient vectors have the same or opposite direction: $\nabla f = \lambda \nabla g$, λ is a constant, called Lagrange Multiplier.

Method of Lagrange Multipliers:

- Solve the system of equations:

$$\nabla f = \lambda \nabla g, \quad g(x, y) = 0.$$

  Let's say, the solutions are (x_i, y_i), $i = 1, ..., n$.

- The maximum = $max\{f(x_i, y_i) : i = 1, ..., n\}$. The minimum = $min\{f(x_i, y_i) : i = 1, ..., n\}$.

Example 34. *Find the max and min of* $z = f(x, y) = xy$, *subject to* $x^2 + y^2 = 1$.

Solution: Here $g(x, y) = x^2 + y^2 - 1$. $\nabla f = (y, x), \nabla g = (2x, 2y)$. So we have

$$\begin{aligned} y &= \lambda 2x, \\ x &= \lambda 2y, \\ x^2 + y^2 &= 1. \end{aligned}$$

We have $x = \pm\frac{1}{\sqrt{2}}$, $y = \pm\frac{1}{\sqrt{2}}$.

The maximum value of z is $z(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = z(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = 1/2$,

and the minimum value of z is $z(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = z(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = -1/2$.

Case 2. Three variables with one constraint: Find the extreme values of the function $w = f(x, y, z)$ subject to constraint $g(x, y, z) = 0$.

Interpretation. Find the extreme values of (x, y, z) on a surface.

When w attains an extreme value a, the level curve $f(x, y, z) = a$ and $g(x, y, z) = 0$ have the same tangent plane. Hence their gradient vectors have the same or opposite direction: $\nabla f = \lambda \nabla g$, λ is a constant, called Lagrange Multiplier.

Method of Lagrange Multipliers:

- Solve the system of equations:

$$\nabla f = \lambda \nabla g, \quad g(x, y, z) = 0.$$

  Let's say, the solutions are (x_i, y_i, z_i), $i = 1, ..., n$.

- The maximum $= max\{f(x_i, y_i, z_i) : \ i = 1, ..., n\}$. The minimum $= min\{f(x_i, y_i, z_i) : i = 1, ..., n\}$.

Example 35. *Find the maximum and minimum of* $w = f(x, y, z) = xyz$ *subject to* $2xz + 2yz + xy - 12 = 0$.

Solution: Here $g(x, y, z) = 2xz+2yz+xy-12$. $\nabla f = (yz, xz, xy), \nabla g = (2z+y, 2z+x, 2x + 2y)$. By

$$\nabla f = \lambda \nabla g, \quad g(x, y, z) = 0,$$

we imply that

$$\begin{aligned} yz &= \lambda(2z + y), \\ xz &= \lambda(2z + x), \\ xy &= \lambda(2x + 2y), \\ 2xz + 2yz + xy - 12 &= 0. \end{aligned}$$

Critical points are: $(x, y, z) = (2, 2, 1), (-2, -2, -1)$.

The maximum value of w is $w = f(2, 2, 1) = 4$; and the minimum value of w is $w = f(-2, 2, -1) = -4$.

Example 36. *Find the maximum and minimum of* $f(x, y, z) = xyz$ *subject to* $x+y+z = 1$.

Solution: Here $g(x, y, z) = x + y + z - 1$. $\nabla f = (yz, xz, xy), \nabla g = (1, 1, 1)$. By

$$\nabla f = \lambda \nabla g, \quad g = 0,$$

we imply that

$$\begin{aligned} yz &= \lambda, & (1) \\ xz &= \lambda, & (2) \\ xy &= \lambda, & (3) \\ x + y + z - 1 &= 0. & (4) \end{aligned}$$

(1)-(2): $z(y - x) = 0$, which implies that $z = 0$ or $x = y$;
(3)-(2): $x(y - z) = 0$, which implies that $x = 0$ or $z = y$.
Next we consider 4 combinations.

- If $z = 0, x = 0$: by (4), $y = 1$, so we get a point (0,1,0);
- If $z = 0, z = y$: then $y = 0$, by (4), $x = 1$, so we get a point (1,0,0);
- If $x = y, x = 0$: then $y = 0$, by (4), $z = 1$, so we get a point (0,0,1);
- If $x = y, z = y$: then by (4), $y = 1/3$, so we get a point (1/3,1/3, 1/3).

$f(1/3, 1/3, 1/3) = 1/27$, which is the maximum;
$f(1, 0, 0) = f(0, 0, 1) = f(0, 1, 0) = 0$, which is the minimum.

Example 37. *Find the max and min values of $w = f(x, y, z) = xyz$ on the sphere $x^2 + y^2 + z^2 = 27$.*

Solution: Here $g(x, y, z) = x^2 + y^2 + z^2 - 27$. $\nabla f = (yz, xz, xy), \nabla g = (2x, 2y, 2z)$. By

$$\nabla f = \lambda \nabla g, \quad g(x, y, z) = 0,$$

we imply that

$$\begin{aligned} yz &= \lambda(2x), \\ xz &= \lambda(2y), \\ xy &= \lambda(2z), \\ x^2 + y^2 + z^2 - 27 &= 0. \end{aligned}$$

If $\lambda = 0$, critical points are: $(\pm 3\sqrt{3}, 0, 0), (0, \pm 3\sqrt{3}, 0), (0, 0, \pm 3\sqrt{3})$.

If $\lambda \neq 0$, critical points are $(\pm 3, \pm 3, \pm 3)$.

The maximum value of w is 8; and the minimum value of w is -8.

Case 3. Three variables with two constraints: Find the extreme values of the function $w = f(x, y, z)$ subject to constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$.

Interpretation. Find extreme values of $f(x, y, z)$ on a 3-D curve.

When w attains an extreme value a, the tangent lines of the curve $g(x, y, z) = 0, h(x, y, z) = 0$, are in the tangent plane of the level surface $f(x, y, z) = a$. Hence, the gradient vector of level surface $f(x, y, z) = a$ and the gradient vectors of $g(x, y, z) = 0, h(x, y, z) = 0$ are in the same plane:

$$\nabla f = \lambda \nabla g + \mu \nabla h,$$

λ and μ are a constants.

Method of Lagrange Multipliers: Solve the system of equations:

$$\nabla f = \lambda \nabla g + \mu \nabla h, \quad g(x, y, z) = 0, h(x, y, z) = 0.$$

For each solution (x, y, z, λ, μ) of this system of equations, find the value of $f(x, y, z)$. The maximum is the maximum value of w, and the minimum is the minimum of w.

Remark. λ, μ are called Lagrange Multipliers.

Example 38. *Find the max and min of $w = f(x, y, z) = x + y + 7z$ subject to $x - y + z = 1, x^2 + y^2 = 1$.*

Solution: Here $g(x, y, z) = x - y + z - 1$, $h(x, y, z) = x^2 + y^2 - 1,$, $\nabla f = (1, 1, 7), \nabla g = (1, -1, 1), \nabla h = (2x, 2y, 0)$. So we have

$$\begin{aligned} 1 &= \lambda + \mu 2x, \\ 1 &= -\lambda + \mu 2y, \\ 7 &= \lambda, \\ x - y + z - 1 &= 0, \\ x^2 + y^2 - 1 &= 0. \end{aligned}$$

We have $(x, y, z) = (-0.6, 0.8, 2.4), (0.6, -0.8, -0.4)$.

The maximum value of w is $w = f(-0.6, 0.8, 2.4) = 17$;

and the minimum value of w is $w = f(0.6, -0.8, -0.4) = -3$.

15.1 Double integrals over rectangles

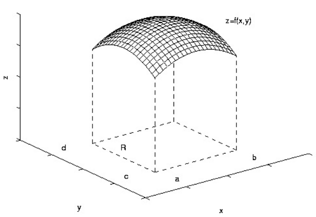
Consider the function $z = f(x, y)$ defined on a rectangle $R : a \le x \le b, c \le y \le d$. Subdivide $[a, b]$ into $a = x_0 < x_1 < \cdots < x_m = b$, and $[c, d]$ into $c = y_0 < y_1 < \cdots < y_n = d$.

The double integral over this rectangle is

$$\iint\limits_R f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A,$$

where $\Delta A = \Delta x \Delta y$, $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$, $x_{i-1} \le x_{ij} \le x_i$, $y_{i-1} \le y_{ij} \le y_i$.

Geometric meaning: If $f(x, y) \ge 0$, then it is the volume of the solid under the graph of $f(x, y)$, above the x-y plane, bounded by R.



The average value of the function defined inside R is

$$f_{ave} = \frac{1}{(b-a)(d-c)} \iint\limits_R f(x, y)dA.$$

Numerical Approximation: Midpoint Rule: To approximate a double integral numerically, we may choose the middle point in each small rectangle as the sample point, or the top right corner (x_i, y_j) as the sample point.

Properties of Double Integrals:

(i) $\iint\limits_R [f(x, y) + g(x, y)]dA = \iint\limits_R f(x, y)dA + \iint\limits_R g(x, y)dA$.

(ii) $\iint\limits_R cf(x, y)dA = c \iint\limits_R f(x, y)dA$.

(iii) If $f(x, y) \le g(x, y)$ for all $(x, y) \in R$, then $\iint\limits_R f(x, y)dA \le \iint\limits_R g(x, y)dA$.

(iv) If $R = R_1 \bigcup R_2$, $R_1 \bigcap R_2 = \emptyset$, then $\iint\limits_R f(x, y)dA = \iint\limits_{R_1} f(x, y)dA + \iint\limits_{R_2} f(x, y)dA$.

(v) $\iint\limits_R dA =$ the area of R.

(vi) If $m \le f(x, y) \le M$ for all $(x, y) \in R$, then $mA \le \iint\limits_R f(x, y)dA \le MA$, where A is the area of R.

Iterated Integrals

Fubini's Theorem: If $f(x,y)$ is continuous on the rectangle $R : a \le x \le b, c \le y \le d$, then

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dy\,dx = \int_c^d \int_a^b f(x,y)dx\,dy.$$

The right hand side is called iterated integral. By this theorem, we can evaluate a double integral using an iterated integral.

Special case: If $f(x,y) = g(x)h(y)$, then the iterated integral becomes the product of two integrals.

$$\iint_R f(x,y)dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$

Example 39. *Find* $\iint_R z dA$*, where* $z = y\sin(xy)$*,* $R = \{(x,y)|1 \le x \le 2, 0 \le y \le \pi/2\}$.

Solution:

$$\iint_R f(x,y)dA = \int_0^{\pi/2}\int_1^2 y\sin(xy)dx\,dy = \int_0^{\pi/2} (-\cos(xy))\,|_1^2 dy$$

$$= \int_0^{\pi/2} (-\cos(2y) + \cos(y))\,dy = \left(-\frac{1}{2}\sin(2y) + \sin(y)\right)|_0^{\pi/2} = 1.$$

Remark. We may use the other order to integrate with respect to x first, but it involves an integral that is harder to evaluate.

Example 40. *Find* $\iint_R z dA$*, where* $z = 16 - x^2 - 2y^2$*,* $R = [0,2] \times [0,2]$.

Solution:

$$\iint_R z dA = \int_0^2\int_0^2 (16 - x^2 - 2y^2)dx\,dy = \int_0^2 \left(16x - \frac{1}{3}x^3 - 2xy^2\right)|_{x=0}^2 dy$$

$$= \int_0^2 \left(\frac{88}{3} - 4y^2\right)dy = 48.$$

15.2 Double integrals over general domains

y**-simple (or Type I):** A region R is of Type I, if

$$R = R_{yx} = \{(x,y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}.$$

Then

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx.$$

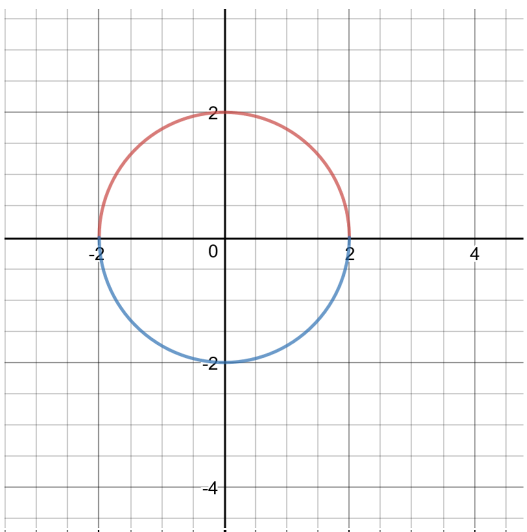
x-simple (or Type II): A region R is of Type II, if

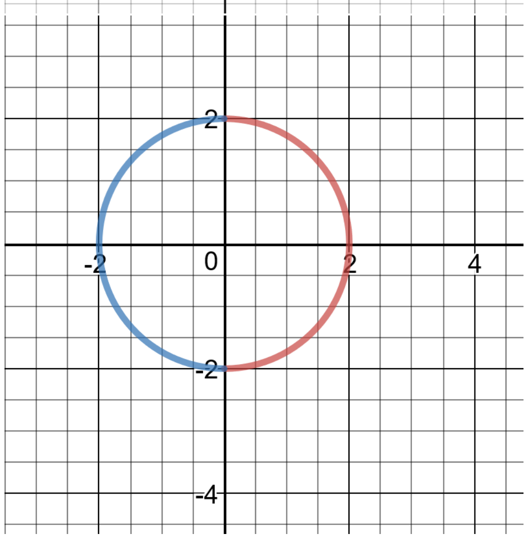
$$R = R_{xy} = \{(x,y) : c \le y \le d, h_1(y) \le x \le h_2(y)\}.$$

Then

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dx\,dy.$$

Example 41. *A region $D: \ x^2+y^2 \le 4$. Rewrite it as y-simple, and x-simple region.*



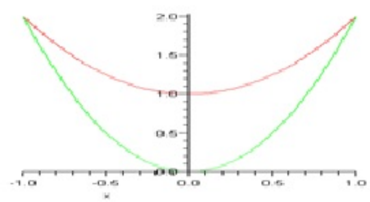


Solution:

- y-simple: $-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$, $-2 \le x \le 2$.

- x-simple: $-\sqrt{4-y^2} \le x \le \sqrt{4-y^2}$, $-2 \le y \le 2$.

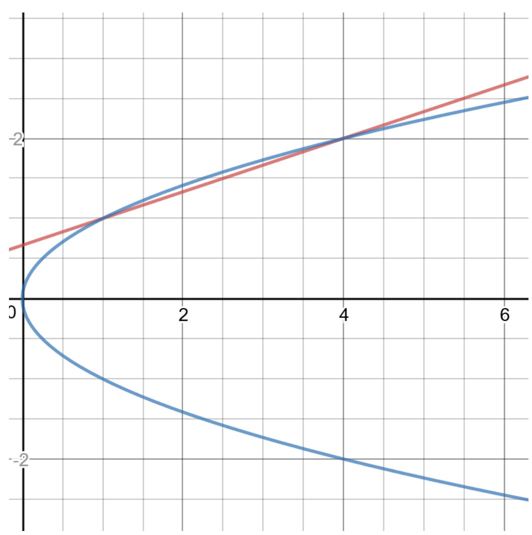
Example 42. *Find* $\iint\limits_R z dA$*, where* $z = x + 2y$*,* R *is the region bounded by* $y = 2x^2$ *and* $y = x^2 + 1$.



Solution: The intersection points of $y = 2x^2$ and $y = x^2 + 1$ are $x = \pm 1$. Thus $R = \{(x, y) : -1 \le x \le 1, 2x^2 \le y \le x^2 + 1\}$.

$$\iint\limits_R z dA = \int_{-1}^{1} \int_{2x^2}^{x^2+1} (x + 2y) dy\, dx = \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + 1)\, dx = \frac{32}{15}.$$

Example 43. *Evaluate* $\iint_D \ln y \, dA$*,* D *is the region bounded by* $3y = x + 2$ *and* $x = y^2$.



Solution: The intersections are $(1, 1), (4, 2)$. Thus $D = R_{xy} = \{(x, y) : 1 \le y \le 2, y^2 \le x \le 3y - 2\}$. Thus

$$\iint_D \ln y \, dA = \int_1^2 \int_{y^2}^{3y-2} \ln y \, dx \, dy$$

$$= \int_1^2 (3y - 2 - y^2) \ln y \, dy$$

$$= \left[(\frac{3}{2}y^2 - 2y - \frac{1}{3}y^3) \ln y - (\frac{3}{4}y^2 - 2y - \frac{1}{9}y^3) \right]_1^2$$

$$= -\frac{2}{3}\ln 2 + \frac{19}{36}.$$

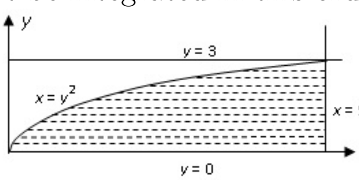
General Region:

A region R has to be subdivided into a number of such regions and calculate the integral separately. If a region can be regarded either as a Type I region, or a Type II region, in some cases, the order of integration is significant.

Changing the order of integration: Some regions can be regarded as of Type I or of Type II. We may use two different ways to express a double integral over such a region as iterated integral. In some cases, both ways are appropriate, and give the same result. However, in some cases, one iterated integral can be evaluated, but the other cannot.

Example 44. *Find* $\int_0^3 \int_{y^2}^9 y\sin(x^2)dxdy$.

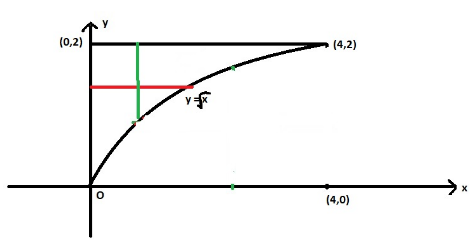
Since the integral $\int_{y^2}^9 y\sin(x^2)dx$ cannot be integrated analytically, this iterated integral cannot be integrated in this order. We need to change the order.



Solution:

$$\int_0^3 \int_{y^2}^9 y\sin(x^2)dxdy = \iint\limits_R y\sin(x^2)dA = \int_0^9 \int_0^{\sqrt{x}} y\sin(x^2)dydx = \frac{1-\cos 81}{4}.$$

Example 45. *Sketch and shade the region of the integral* $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \sqrt{1+y^3}\ dydx$*, then evaluate the integral.*



Solution:

$$\{(x,y) : \sqrt{x} \le y \le 2, 0 \le x \le 4\} \to \{(x,y) : 0 \le y \le 2, 0 \le x \le y^2\}$$

$$\int_0^4 \int_{\sqrt{x}}^2 \sqrt{1+y^3}\ dydx = \int_0^2 \int_0^{y^2} \sqrt{1+y^3}\ dx\ dy$$

$$= \int_0^2\ y^2\sqrt{1+y^3}\ dy$$

$$\int_1^9 \frac{1}{3}\ \sqrt{u}\ du, \quad u = 1+y^3$$

$$= \frac{2}{9}u^{3/2}|_1^9 = \frac{52}{9}.$$

15.3 Double integrals in polar coordinates

Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$. A polar rectangle is defined to be $R = \{(r,\theta) : \alpha \le \theta \le \beta, a \le r \le b\}$, where $0 \le \beta - \alpha \le 2\pi$.

The double integral over a polar rectangle R is evaluated by the iterated integral:

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta, \quad dA = rdrd\theta.$$

Example 46. *Find* $\iint_R zdA$, *where* $z = e^{-(x^2+y^2)}$, $R = \{(x,y) : x \ge 0, y \ge 0, x^2+y^2 \le a^2\}$.

Solution: In polar coordinates, $S = \{(r,\theta) : 0 \le \theta \le \pi/2, 0 \le r \le a\}$

$$\iint_R zdA = \int_0^{\pi/2} \int_0^a e^{-(r^2)}rdrd\theta = \frac{(1-e^{-a^2})\pi}{4}.$$

If $R = \{(r,\theta) : \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$, then

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta.$$

Example 47. *Find the volume of the solid above the x-y plane, under the paraboloid* $z = x^2+y^2$, *and inside the cylinder* $x^2+y^2-2x = 0$.

Solution: The base of the cylinder is the circle in xy-plane $(x-1)^2+y^2 = 1$ with center (1, 0) and radius 1. In polar coordinate system, the equation of the circle is $r = 2\cos\theta$. Thus

$$R = \{(r,\theta) : -\pi/2 \le \theta \le \pi/2, 0 \le r \le 2\cos\theta\}.$$

$$V = \iint_R zdA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2rdrd\theta = \int_{-\pi/2}^{\pi/2} 4\cos^4\theta d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} + 2\cos(2\theta) + \frac{1}{2}\cos(4\theta)\right) d\theta = \frac{3\pi}{2}.$$

15.4 Applications of Double Integrals

Volume of solids under a surface: Let $z = f(x,y)$, $(x,y) \in R$ define a surface S, where $f(x,y) \geq 0$ for all $(x,y) \in D$. Then the volume V of the solid lying directly above D is:

$$V = \iint_D f(x,y)dA.$$

Example 48. *Find the volume under the surface $z = 4 - x^2 - y^2$ that projects onto the region*

$$R = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Solution:

$$V = \iint_R zdA = \int_0^1 \int_0^1 (4 - x^2 - y^2)dx\,dy == \int_0^1 (4 - x^2 - \frac{1}{3})dx = 4 - \frac{2}{3} = \frac{10}{3}.$$

Example 49. *Find the volume of the solid bounded by $z = \sin x \cos y$, the planes $x = \pi/2$ and $y = \pi/2$, and the three coordinates.*

Solution:

$$V = \iint_R zdA = \int_0^{\pi/2} \int_0^{\pi/2} \sin x \cos y dx\,dy = 1.$$

Example 50. *Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.*

Solution: Let $R = \{(x,y) : x^2 + y^2 \leq a^2\}$. Then

$$V = 2\iint_R z\ dA = 2\iint_R \sqrt{a^2 - x^2 - y^2}\ dA$$

$$= 2\iint_D \sqrt{a^2 - r^2} r d\theta dr = 2\int_0^a \int_0^{2\pi} r\sqrt{a^2 - r^2} d\theta dr$$

$$= 4\pi \int_0^a r\sqrt{a^2 - r^2} dr = -2\pi \int_{a^2}^0 \sqrt{u} du, \quad u = a^2 - r^2$$

$$= -2\pi \frac{2}{3} u^{3/2}|_{a^2}^0 = \frac{4}{3}\pi a^3.$$

Mass of the lamina: Let $z = \rho(x,y)$, $(x,y) \in D$ define the density of the lamina occupies the region D. Then the mass m of the lamina is:

$$m = \iint_D \rho(x,y)dA.$$

Example 51. *Find the mass of the lamina that occupies the region* $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ *with the density function* $\rho(x, y) = xye^{x^2}$.

Solution:

$$m = \iint_D \rho(x, y)dA = \int_0^1 \int_0^1 xye^{x^2}\ dx\ dy$$

$$\int_0^1 \left[\frac{1}{2}ye^{x^2}\right]_{x=0}^1\ dy = \int_0^1 \frac{e-1}{2}y\ dy = \frac{e-1}{4}.$$

Moments and Centers of Mass: Now lets find the center of mass of a lamina with density function $\rho(x, y)$ that occupies a region D.

Recall that the moment of a particle about an axis is defined as the product of its mass and its directed distance from the axis. The moments of the entire lamina about the x-axis and about the y-axis are:

$$M_x = \iint_D y\rho(x, y)dA, \quad M_y = \iint_D x\rho(x, y)dA.$$

The center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ and mass m are:

$$(\bar{x}, \bar{y}) = (\frac{M_y}{m}, \frac{M_x}{m}).$$

Example 52. *Find the center of mass of a quarter circle of radius* r*, assuming that the density throughout is uniformly constant.*

Solution: Assume $\rho(x, y) = 1$. $m = \pi r^2/4$, $M_y = r^3/3$, $M_x = r^3/6$, $(\bar{x}, \bar{y}) = (\frac{4r}{3\pi}, \frac{4r}{3\pi})$.

Example 53. *Let* $\rho(x, y) = x^2+y^2$*, and* D *is a triangle bounded by* $x = 0$*,* $x = y$*,* $x+y = 2$*. Find the mass of the lamina and the center of mass.*

Solution: $m = \frac{4}{3}$, $M_y = \frac{7}{15}$, $M_x = \frac{5}{3}$, $(\bar{x}, \bar{y}) = (\frac{7}{20}, \frac{5}{4})$.

15.5 Surface area

Surface area: For a differentiable surface S: $z = f(x, y)$, $(x, y) \in D$, the area of the surface is

$$\iint_S dS = \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\ dA.$$

$$dS = \sqrt{(f_x)^2 + (f_y)^2 + 1}\ dA.$$

Example 54. *Find the area of the surface $z = y^2 + x$ that lies directly above the region $\{0 \le x \le \sqrt{2}, x \le y \le \sqrt{2}\}$.*

Solution: Change the order: $\{0 \le x \le y, 0 \le y \le \sqrt{2}\}$.

$$A = \int_0^{\sqrt{2}} \int_x^{\sqrt{2}} \sqrt{2 + 4y^2} dy dx = \int_0^{\sqrt{2}} \int_0^{y} \sqrt{2 + 4y^2} dx dy$$

$$= \int_0^{\sqrt{2}} y\sqrt{2 + 4y^2} dy = \frac{1}{8} \int_2^{10} \sqrt{u} du, \quad u = 2 + 4y^2.$$

Example 55. *Find the surface area of the sphere $x^2 + y^2 + z^2 = a^2$.*

Solution: Let $R = \{(x, y) : x^2 + y^2 \le a^2\}$, $z = \sqrt{a^2 - x^2 - y^2}$. Then

$$z_x = -\frac{x}{\sqrt{a^2 - x^2 - y^2}}, \quad z_y = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}.$$

$$surface\ area = 2 \iint_R \sqrt{z_x^2 + z_y^2 + 1}\ dA = 2 \iint_R \frac{a}{\sqrt{a^2 - x^2 - y^2}}\ dA$$

$$= 2 \iint_D \frac{a}{\sqrt{a^2 - r^2}} r d\theta dr, \quad x = r\cos\theta, y = r\sin\theta$$

$$= 2a \int_0^a \int_0^{2\pi} \frac{r}{\sqrt{a^2 - r^2}} d\theta dr = 4\pi a \int_0^a \frac{r}{\sqrt{a^2 - r^2}} dr$$

$$= -2\pi a \int_{a^2}^0 u^{-1/2} du, \quad u = a^2 - r^2$$

$$= 4\pi a^2.$$

15.6 Triple Integrals, Applications

Triple integral of $f(x,y,z)$ over the solid E:

$$\iiint_E f(x,y,z)dV.$$

- If $f=1$, the triple integral is **the volume** of the solid E.
- If f is the density function, the triple integral gives **the mass** of E.

Triple Integral over a Rectangular Box

A rectangular box is the region in 3-dimensional space defined by

$$B=\{(x,y,z): a\le x\le b, c\le y\le d, r\le z\le s\}.$$

Fubini's Theorem: If $f(x,y,z)$ is continuous on the rectangular box B, then

$$\iiint_B f(x,y,z)dV=\int_r^s\int_c^d\int_a^b f(x,y,z)dx\,dy\,dz.$$

This integral can also be evaluated by the other orders of the variables.

Example 56. *Find the mass of $B=\{(x,y,z): 0\le x\le 2, 0\le y\le 2, 0\le z\le 2\}$ with the density function*

$$\delta(x,y,z)=x+y+z.$$

$$M=\iiint_B \delta(x,y,z)dV=\iiint_B (x+y+z)dV$$

$$=\int_0^2\int_0^2\int_0^2(x+y+z)dx\,dy\,dz=\int_0^2\int_0^2 (\frac{1}{2}x^2+xy+xz)\Big|_{x=0}^2 dy\,dz$$

$$=\int_0^2\int_0^2(2y+2z+2)\,dy\,dz=\int_0^2 (y^2+2yz+2y)\big|_{y=0}^2\,dz=\int_0^2(4z+8)\,dz=24.$$

Triple Integrals over a General Region

z-simple region (Type I). The region is bounded by a cylinder $F(x,y)=0$, and the graphs of two functions of x and y:

$$E=\{(x,y,z):(x,y)\in D, u_1(x,y)\le z\le u_2(x,y)\},$$

where D is the region in (x, y) plane bounded by the graph of $F(x, y) = 0$. A triple integral over a region E of type I is evaluated by

$$\iiint_E f(x,y,z)dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz \right) dA.$$

x-simple region (Type II). The region is bounded by a cylinder $F(y, z) = 0$, and the graphs of two functions of y and z:

$$E = \{(x, y, z) : (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\},$$

where D is the region in (y, z) plane bounded by the graph of $F(y, z) = 0$. A triple integral over a region E of type II is evaluated by

$$\iiint_E f(x,y,z)dV = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx \right) dA.$$

y-simple region (Type III). The region is bounded by a cylinder $F(x, z) = 0$, and the graphs of two functions of x and z:

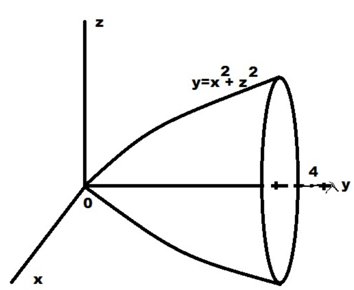
$$E = \{(x, y, z) : (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\},$$

where D is the region in (x, z) plane bounded by the graph of $F(x, z) = 0$. A triple integral over a region E of type III is evaluated by

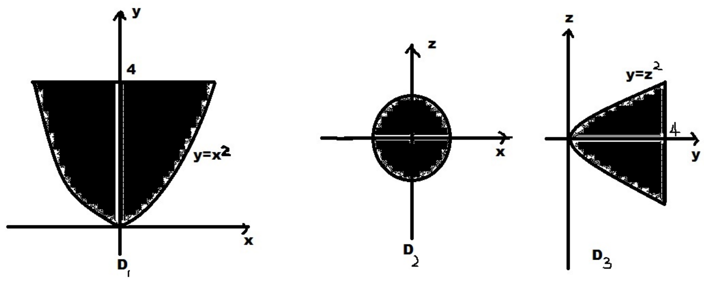
$$\iiint_E f(x,y,z)dV = \iint_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy \right) dA.$$

Remark. If a region is not of any of these types, we can subdivide this region into a finite number of regions of these types. The triple integral is the sum of triple integrals over sub-regions.

Example 57. *Describe the region E by z-simple, x-simple, and y-simple respectively, where E is the region bounded by paraboloid $y = x^2 + z^2$ and $y = 4$.*



The projections are:



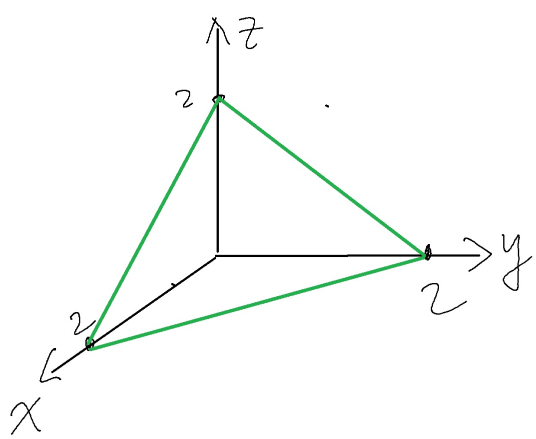
Solution:

- z**-simple:** $-\sqrt{y-x^2} \leq z \leq \sqrt{y-x^2}$, $x^2 \leq y \leq 4$, $-2 \leq x \leq 2$.

- y**-simple:** $x^2+z^2 \leq y \leq 4$, $-\sqrt{4-x^2} \leq z \leq \sqrt{4-x^2}$, $-2 \leq x \leq 2$.

- x**-simple:**$-\sqrt{y-z^2} \leq x \leq \sqrt{y-z^2}$, $z^2 \leq y \leq 4$, $-2 \leq z \leq 2$.

Example 58. *Evaluate* $\iiint\limits_E (x+y)dV$ *, where* E *is the tetrahedron bounded by planes* $x=0, y=0, z=0,$ *and* $x+y+z=2$.



Solution: We consider E as z-simple region. Then the projection to xy-plane is the triangle bounded by $x=0$, $y=0$ and $x+y=2$. Thus

$$0 \leq z \leq 2-x-y, 0 \leq y \leq 2-x, 0 \leq x \leq 2.$$

$$\iiint\limits_E (x+y)dV = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} (x+y)\ dzdydx$$

$$= \int_0^2 \int_0^{2-x} (x+y)(2-x-y)\ dydx$$

$$= \int_0^2 \int_0^{2-x} (2x - x^2 - 2xy + 2y - y^2)\ dydx$$

$$= \int_0^2 \left[(2x - x^2)(2-x) - x(2-x)^2 + (2-x)^2 - \frac{1}{3}(2-x)^3 \right]\ dx$$

$$= \int_0^2 \left[(2-x)^2 - \frac{1}{3}(2-x)^3 \right]\ dx$$

$$= -\int_2^0 \left[u^2 - \frac{1}{3}u^3 \right]\ dx, \qquad u = 2-x$$

$$= (\frac{1}{3}u^3 - \frac{1}{12}u^4)|_0^2 = \frac{4}{3}.$$

15.7 Triple integrals in cylindrical coordinates.

Cylindrical coordinate system uses (r, θ, z) to specify a point in space, where r and θ are polar coordinates of the projection of the point on the xy-plane.

$$x = r\cos\theta, y = r\sin\theta, z = z; \quad r = \sqrt{x^2+y^2}, \cos\theta = \frac{x}{r}, \sin\theta = \frac{y}{r}, z = z.$$

Example 59. $(r, \theta, z) = (-1, \pi, 2) \Rightarrow (x, y, z) = (1, 0, 2)$; $(x, y, z) = (\sqrt{3}, -1, 2) \Rightarrow (r, \theta, z) = (2, -\pi/6, 2)$.

Suppose the region E of integration is of type I. A triple integral can be evaluated with the cylindrical coordinates:

$$x = r\cos\theta, y = r\sin\theta, z = z, \quad dV = rdzdrd\theta.$$

Let D be the projection of E onto the xy-plane. Then

$$\iiint\limits_E f(x,y,z)dV = \iint\limits_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz\, dA$$

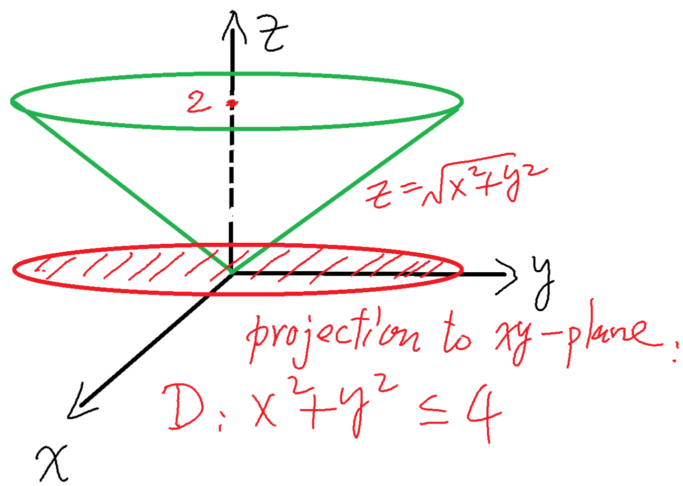
$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r dzdrd\theta.$$

Example 60. *Find the integral* $\iiint\limits_E (x^2+y^2)dV$ *, where E is the solid bounded by*

$$-2 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \sqrt{x^2+y^2} \le z \le 2.$$

Remark. The question is equivalent to the integral

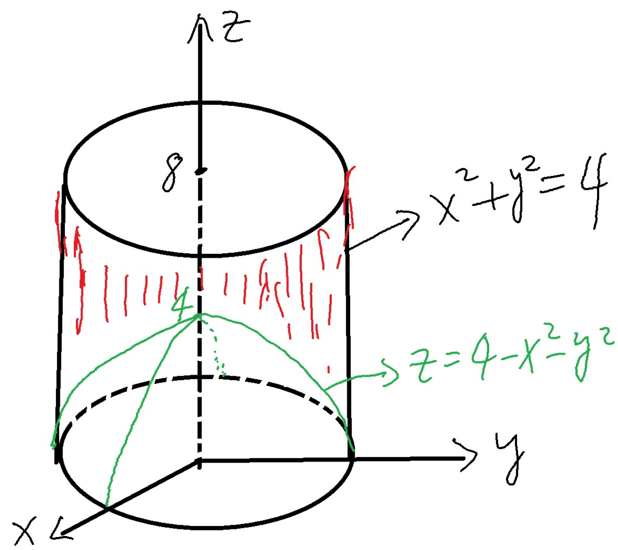
$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\;dz\;dy\;dx.$$



Solution: Note that E is the region bounded by the cone $z=\sqrt{x^2+y^2}$ and the plane $z=2$. In cylindrical coordinates, we obtain $0\le r\le 2$, $0\le\theta\le 2\pi$, $r\le z\le 2$. Thus

$$\iiint\limits_E (x^2+y^2)dV=\int_0^{2\pi}\int_0^2\int_r^2 r^2 r\,dz\,dr\,d\theta=\frac{16\pi}{5}.$$

Example 61. *Find the integral* $\iiint\limits_E zdV$, *where E is within* $x^2+y^2=4$, *below* $z=8$, *above* $z=4-x^2-y^2$.



Solution: By cylindrical coordinates,

$$E = \{0 \le r \le 2, 0 \le \theta \le 2\pi, 4 - r^2 \le z \le 8\}.$$

$$\iiint_E z dV = \int_0^{2\pi} \int_0^2 \int_{4-r^2}^8 z\ rdz\ dr\ d\theta$$

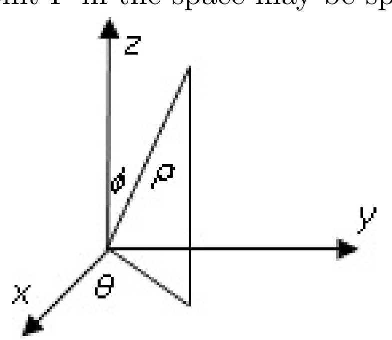
$$= \int_0^{2\pi} \int_0^2 \frac{1}{2} r z^2 |_{z=4-r^2}^8\ dr\ d\theta$$

$$= \int_0^{2\pi} \int_0^2 \frac{1}{2} r(48 + 8r^2 - r^4)\ dr\ d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \left(24r^2 + 2r^4 - \frac{1}{6} r^6\right) |_{r=0}^2\ d\theta = \frac{352}{3}\pi$$

15.8 Triple Integrals in Spherical Coordinates

A point P in the space may be specified (ρ, θ, ϕ):



$$x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho\cos\phi, dV = \rho^2 \sin\phi d\rho d\phi d\theta,$$

where $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho^2 sin\phi$.

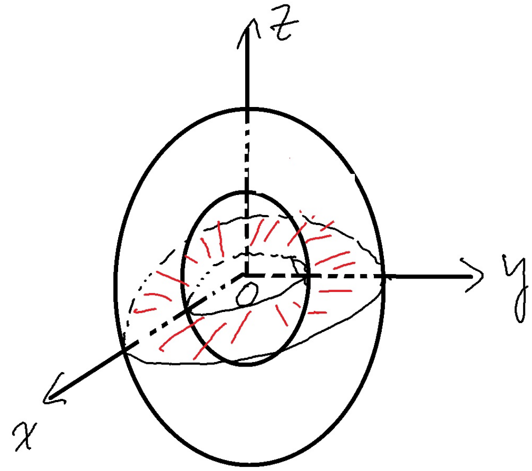
Example 62. *$(x, y, z) = (-\sqrt{3}, 1, 2)$, find (ρ, θ, ϕ).*

If a region E is specified in spherical coordinates, then

$$\iiint_E f(x, y, z) dV = \iiint_E f(\rho sin\phi \cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi d\rho d\phi d\theta,$$

where the order of integration depends on the definition of E.

Example 63. *Find the integral $I = \iiint_E \sqrt{x^2 + y^2 + z^2} dV$, where E is region between the sphere $x^2 + y^2 + z^2 = 1$, and the sphere $x^2 + y^2 + z^2 = 4$.*

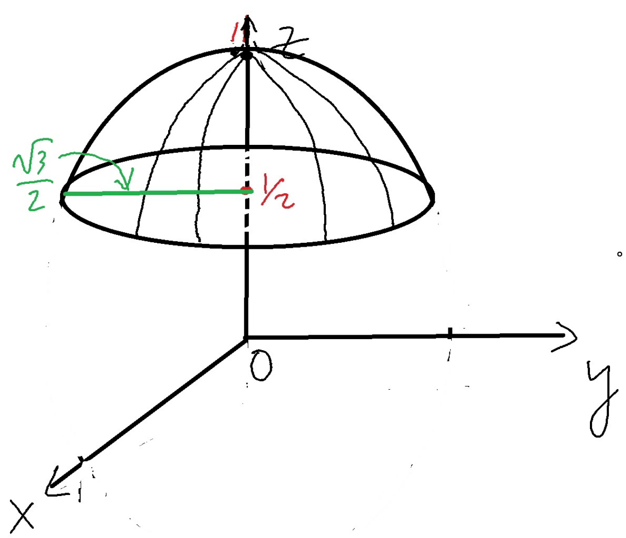


Solution: It is easy to see that $1 \le \rho \le 2, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi$. Thus

$$I = \iiint\limits_E \sqrt{x^2+y^2+z^2}dV = \int_0^{2\pi}\int_0^{\pi}\int_1^2 \sqrt{\rho^2}\rho^2 \sin\phi d\rho d\phi d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi \, d\phi\right)\left(\int_1^2 \rho^3 d\rho\right) = 15\pi.$$

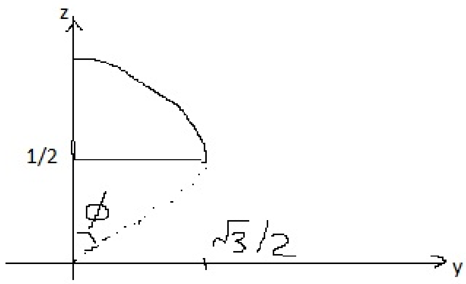
Example 64. *Find the integral* $I = \iiint\limits_E zdV$*, where* E *is region within the sphere* $x^2 + y^2 + z^2 = 1$ *and above the plane* $z = 1/2$.



Solution: The intersection of the sphere $x^2+y^2+z^2 = 1$ and the plane $z = 1/2$ is

$$x^2 + y^2 = 3/4, \Rightarrow 0 \le \theta \le 2\pi.$$

To find the interval for ϕ, we look at the intersection of the solid and the yz-plane:



$$y^2 = 3/4 \Rightarrow y = \sqrt{3}/2 \Rightarrow \tan\phi = \frac{\sqrt{3}/2}{1/2} = \sqrt{3} \Rightarrow \phi = \pi/3 \Rightarrow 0 \le \phi \le \pi/3.$$

$$z \ge 1/2 \Rightarrow \rho\cos\phi \ge 1/2 \Rightarrow \rho \ge \frac{1}{2\cos\phi}.$$

$$x^2 + y^2 + z^2 \le 1 \Rightarrow \rho^2 \le 1 \Rightarrow \rho \le 1.$$

Thus

$$I = \iiint\limits_E z dV = \int_0^{2\pi} \int_0^{\pi/3} \int_{1/(2\cos\phi)}^{1} \rho\cos\phi\rho^2 \sin\phi d\rho d\phi d\theta$$

$$= \frac{1}{64} \int_0^{2\pi} \int_0^{\pi/3} \sin\phi\cos\phi \left(16 - \frac{1}{\cos^4\phi}\right) d\phi d\theta = \frac{9\pi}{64}.$$

15.9 Change of variables in multiple integrals.

Change of Variables for a Triple Integral:

Change of Variables for a Double Integral: Suppose that we want to integrate $f(x, y)$ over the region R. Under the transformation $T : x = g(u, v), y = h(u, v)$, the region R becomes S, and the integral becomes,

$$\iint\limits_R f(x, y) dA = \iint\limits_S f(g, h) J(u, v) du dv,$$

where

- $J(u, v) = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$, and is called the Jacobian of the transformation.
- $dA = J(u, v) du dv$.
- For polar coordinates $x = r\cos\theta,\ y = r\sin\theta,\ J(r, \theta) = r$.

- For elliptic coordinates $x = ar\cos\theta,\ y = br\sin\theta,\ J(r,\theta) = abr$.

Example 65. *Use the change of variables $x = 2u+3v$, $y = 2u-3v$ to evaluate the integral $\iint\limits_R (x+y)dA$ where R is the trapezoidal region with vertices given by (0,0), (5,0), (5/2, 5/2), (5/2, -5/2).*

Solution: The trapezoidal region R is bounded by four lines: $y = x, y = -x+5, y = x-5, y = -x$. After the transformation , we get a rectangle S: $u = 0, v = 0, u = 5/4, v = 5/6$. The Jacobian is -12. Thus

$$\iint\limits_R (x+y)dA = \frac{125}{4}.$$

Example 66. *Find $\iint\limits_R xydA$, where $R = \{(x,y) : x^2 + \frac{y^2}{36} \le 1, x \ge 0, y \ge 0\}$.*

Solution: Let $x = r\cos\theta, y = 6r\sin\theta$. $0 \le r \le 1$, $0 \le \theta \le \pi/2$.

$$\iint_R xydA = \int_0^{\pi/2}\int_0^1 r^2\cos\theta\sin\theta\ 6rdrd\theta$$

$$= 6\left(\frac{1}{4}r^4\right)\bigg|_0^1 \left(\frac{1}{2}\sin^2\theta)\right|_0^{\pi/2} = \frac{3}{4}.$$

Change of Variables for a Triple Integral: Suppose that we want to integrate $f(x,y,z)$ over the region R. Under the transformation $T : x = g(u,v,w), y = h(u,v,w), z = k(u,v,w)$ the region R becomes S, and the integral becomes,

$$\iiint\limits_R f(x,y,z)dV = \iiint\limits_S f(g,h,k)\frac{\partial(x,y,z)}{\partial(u,v,w)}dudvdw,$$

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

and is called the Jacobian of the transformation T, and

$$dV = \frac{\partial(x,y,z)}{\partial(u,v,w)}dudvdw.$$

- Cylindrical coordinates: $J(r,\theta,z) = r$.

- Spherical coordinates: $J(\rho,\phi,\theta) = \rho^2\sin\phi$.

Example 67. *Find the volume of the solid R bounded by the coordinate planes and the surface* $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$.

Solution: Let $x = u^2, y = v^2, z = w^2$. After the transformation , the solid R is changed to S, which is bounded by the plane u+v+w=1 and the three planes u=0, v=0, w=0. The Jacobian is $8uvw$. Thus

$$V = \iiint_R dV = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 8uvw\, dwdvdu = \frac{1}{90}.$$

16.1 Vector fields.

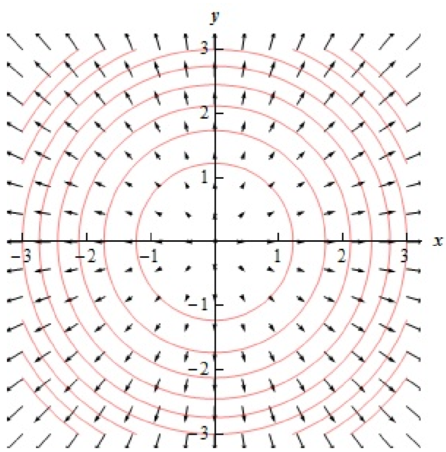
Vector Fields

In a two-dimensional space,vector function $\vec{F}(x,y) = (P(x,y), Q(x,y)) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is a 2-dimensional vector field.

In a three-dimensional space, $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z)) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ is a 3-dimensional vector field.

Vector fields can be visualized by diagrams.

Example 68. *Gradient Vector Fields: Let $f(x,y) = x^2 + y^2$. Then $\vec{F}(x,y) = \nabla f(x,y) = (x,y)$ is a vector field.*



Conservative Vector Fields

A vector field $\vec{F}$ is conservative if there exists a function f such that $\vec{F} = \nabla f$. In other words, a vector field is conservative if it is the gradient field of a (scalar) function. The function f is called a **potential function** of $\vec{F}$.

Remark. The potential function of a conservative vector field is not unique.

Example 69. *(a) Verify that $\vec{F} = (y\cos x, \sin x)$ is a conservative vector field with potential function $f(x,y) = y\sin x$.*

(b) Verify that $\vec{F} = (y^2, 2xy + e^{3z}, 3ye^{3z})$ is a conservative vector field with potential function $f(x,y,z) = xy^2 + ye^{3z}$.

Solution: (a) $\nabla f = (f_x, f_y) = (y\cos x, \sin x)$. Thus $\vec{F} = \nabla f$.

(b) $\nabla f = (f_x, f_y, f_z) = (y^2, 2xy + e^{3z}, 3ye^{3z})$. Thus $\vec{F} = \nabla f$.

A Necessary and Sufficient Condition for a Vector Field to be Conservative:

Let $\vec{F} = (P, Q)$ be a vector field in a simply-connected region D. Suppose P and Q have continuous first-order derivative in D, then $\vec{F}$ is conservative if and only if

$$P_y = Q_x.$$

Example 70. *(i) Show that the vector field* $\vec{F} = (3 + 2xy, x^2 - 3y^2)$ *is conservative.*
(ii) Find a potential function of this field.

Solution: (i) Let $P = 3 + 2xy$, $Q = x^2 - 3y^2$. Since $P_y = 2x = Q_x$, $\vec{F}$ is conservative.
(ii) Let $f(x, y)$ be a potential function. Then $F = (f_x, f_y) = (P, Q)$,

$$f_x = 3 + 2xy, f_y = x^2 - 3y^2.$$

$$f_x = 3 + 2xy \Rightarrow f(x, y) = 3x + x^2y + g(y) \Rightarrow f_y = x^2 + g'(y) \Rightarrow$$

$$x^2 - 3y^2 = x^2 + g'(y), \Rightarrow g'(y) = -3y^2 \Rightarrow g(y) = -y^3.$$

Hence,

$$f(x, y) = 3x + x^2y - y^3.$$

Example 71. *Find the potential function* $f(x, y, z)$ *of the vector fields:* $\vec{F} = (z, 2yz, x+y^2)$.

Solution: (a) Let $f(x, y, z)$ be a potential function. Then $F = (f_x, f_y, f_z) = (z, 2yz, x + y^2)$,

$$f_x = z, f_y = 2yz, f_z = x + y^2.$$

$$f_x = z \Rightarrow f = xz + g(y, z) \Rightarrow$$

$$f_y = g_y = 2yz \Rightarrow g(y, z) = y^2z + h(z), f = xz + y^2z + h(z) \Rightarrow$$

$$f_z = x + y^2 + h'(z) = x + y^2, \Rightarrow h'(z) = 0 \Rightarrow h(z) = constant, C.$$

Hence,

$$f = xz + y^2z + C.$$

Example 72. *Let* $f(x, y, z) = xyz \ln z$ *be a potential function of* $\vec{F}$. *Find* $\vec{F}$.

Solution:

$$\vec{F} = (f_x, f_y, f_z) = (yz \ln z, xz \ln z, xy \ln z + xy).$$

16.2 Line integrals.

- Line integral $\int_C f(x,y)ds$: The area of a "fence" with C as the base, and the height is given by $f(x,y)$.

- The mass of the wire C, if f is the density of the wire.

1. Line Integrals of Scalar fields in 2-D

Let C be a smooth curve given by $x = x(t), y = y(t), a \le t \le b$, or equivalently, by the vector equation $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$. Let the density at a point (x,y) on C be $f(x,y)$. Subdivide this curve segment into a number of small segments. The weight of a small segment of the curve is approximately $f(x^*,y^*)\Delta s$, where (x^*,y^*) is a point in this segment, and Δs is the length of this small segment. The sum $\sum f(x^*,y^*)\Delta s$ is an approximation of the total weight of the curve segment. The total weight of C is $\lim_{\Delta s \to 0} \sum f(x^*,y^*)\Delta s$.

Definition 3. *If f is defined on a smooth curve C, then the line integral of f along C is*

$$\int_C f(x,y)ds = \lim_{\Delta s \to 0} \sum f(x^*,y^*)\Delta s$$

if the limit exists.

This is also called the line integral of type I.

An interpretation of line integral: The area of a "fence" with C as the base, and the height is given by $f(x,y)$.

Calculation of a line integral: If the smooth curve C is defined by parametric equations $x = x(t), y = y(t), a \le t \le b$, then

$$\int_C f(x,y)ds = \int_a^b f(x,y)\sqrt{[x'(t)]^2 + [y'(t)]^2}dt, \quad ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$$

If C can be subdivide into a finite number of segments: $C = C_1 \bigcup C_2 \bigcup \cdots \bigcup C_n$, and smooth on each segment, then the line integral is calculated for each segment and the sum is the line integral of C:

$$\int_C f ds = \int_{C_1} f ds + \cdots + \int_{C_n} f ds.$$

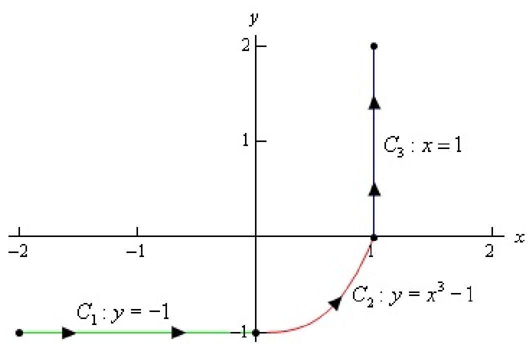
Example 73. *Find the mass of C with density $f(x,y) = y$, where C is the cycloid $x = t - \sin t, y = 1 - \cos t$, $0 \le t \le 2\pi$.*

Solution:

$$\int_C y ds = \int_0^{2\pi} y(t)\sqrt{[x'(t)]^2+[y'(t)]^2}dt = \int_0^{2\pi}(1-\cos t)\sqrt{2-2\cos t}dt$$

$$= \int_0^{2\pi} 4\sin^3(\frac{t}{2})dt = -8\left[\cos(t/2)-\frac{1}{3}\cos^3(t/2)\right]_0^{2\pi} = \frac{32}{3}.$$

Example 74. *Evaluate $\int_C 4x^3 ds$, where C is the curve shown below:*



Solution: The three curves are:

$C_1 : x=t, y=-1, -2 \le t \le 0;\quad C_2 : x=t, y=t^3-1, 0\le t\le 1;\quad C_3 : x=1, y=t, 0\le t\le 2.$

$$\int_{C_1} 4x^3 ds = \int_{-2}^{0} 4t^3\sqrt{1^2+0^2}dt = -16,$$

$$\int_{C_2} 4x^3 ds = \int_0^1 4t^3\sqrt{1^2+(3t^2)^2}dt = \frac{2}{27}(10^{3/2}-1),$$

$$\int_{C_3} 4x^3 ds = \int_0^2 4(1)^3\sqrt{0^2+1^2}dt = 8,$$

thus

$$\int_C 4x^3 ds = \int_{C_1} 4x^3 ds + \int_{C_2} 4x^3 ds + \int_{C_3} 4x^3 ds = \frac{2}{27}(10^{3/2}-1)-8.$$

Special cases: $ds = dx, or ds = dy$: $\int_C f(x,y)dx$ and $\int_C f(x,y)dy$ are called respectively line integral of f along C with respect to x and y. Suppose C is defined by parametric equations $x = u(t), y = v(t)$, $a \le t \le b$, then

$$\int_C P(x,y)dx + Q(x,y)dy = \int_a^b [P(u,v)u'(t) + Q(u,v)v'(t)]dt.$$

Example 75. *Find $I = \int_C (x+y)dx + (x-y)dy$, where C is a curve defined by $x = e^t\sin t, y = e^t\cos t, 0 \le t \le \pi/2$.*

Solution:

$$I=\int_C (x+y)dx+(x-y)dy=\int_0^{\pi/2}[(e^t\sin t+e^t\cos t)(e^t\sin t)'+(e^t\sin t-e^t\cos t)(e^t\cos t)']dt$$

$$=\int_0^{\pi/2}2e^{2t}\sin(2t)dt=\int_0^{\pi}e^w\sin(w)dw=\frac{1}{2}e^w(\sin w-\cos w)=\frac{1}{2}(e^\pi+1).$$

2. Line Integrals of Scalar fields in 3-D

Calculation of a line integral: If the smooth curve C is defined by parametric equations $x=x(t), y=y(t), z=z(t), a\le t\le b$, then

$$\int_C f(x,y,z)ds=\int_a^b f(x,y,z)\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}dt.$$

Example 76. *Find* $\int_C (xy-z)ds$*, where* C *is the cycloid* $x=t, y=t^2, z=\frac{2}{3}t^3$*,* $0\le t\le 1$ *.*

Solution:

$$\int_C (xy-z)ds=\int_0^1 (xy-z)\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}dt=\int_0^1\frac{1}{3}t^3\sqrt{1+4t^2+4t^4}dt=\frac{7}{36}.$$

3. Line Integrals of Vector fields

Definition 4. *Let* $\vec{F}$ *be a continuous vector field defined on a smooth curve* $C:\vec{r}=\vec{r}(t), a\le t\le b$*. Then the line integral of* $\vec{F}$ *along* C *is:*

$$\int_C \vec{F}\cdot d\vec{r}=\int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)dt=\int_C \vec{F}\cdot\vec{T}ds,$$

which can be interpreted as the total work done by this force vector field when this object is moving from one end of C to the other end of C.

$$\int_C \vec{F}\cdot d\vec{r}=\int_C P(x,y)dx+Q(x,y)dy,\ \ if\ \ \vec{F}=(P(x,y),Q(x,y));$$

$$\int_C \vec{F}\cdot d\vec{r}=\int_C P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz,$$

if $\vec{F}=(P(x,y,z),Q(x,y,z),R(x,y,z))$.

Example 77. *Find* $\int_C \vec{F}\cdot d\vec{r}$*, where* $\vec{F}=(x,y,z)$ *and* C *is:* $x=\cos t, y=\sin t, z=\sin t$*,* $0\le t\le\pi/2$ *.*

Solution: By the curve C,

$$\vec{r}(t) = (x, y, z) = (\cos t, \sin t, \sin t), \quad 0 \le t \le \pi/2.$$

$$\vec{F}(\vec{r}(t)) = (x, y, z) = (\cos t, \sin t, \sin t)$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C xdx + ydy + zdz = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)dt$$

$$= \int_0^{\pi/2} [(\cos t)(\cos t)' + (\sin t)(\sin t)' + (\sin t)(\sin t)']dt$$

$$= \int_0^{\pi/2} \sin t \cos t dt = \frac{1}{2}(\sin t)^2|_0^{\pi/2} = \frac{1}{2}.$$

16.3 Fundamental theorem for line integrals.

Fundamental Theorem for Line Integral: Let $\vec{F} = \nabla f$ be a conservative vector field, and C be a smooth curve defined by parametric equation $C : \vec{r} = \vec{r}(t), a \le t \le b$. Then

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Path Independence of Line Integrals: Let $\vec{F}$ be a vector field continuous in an open connected region. The following statements are equivalent:
(a) The line integral is independent of path.
(b) The line integral is zero along any closed curve.
(c) $\vec{F}$ is a conservative vector field.

Example 78. *The vector field* $\vec{F} = (3+2xy, x^2-3y^2)$ *is conservative. A potential function of this field is* $f(x, y) = 3x + x^2y - y^3$. *Find* $\int_C (3 + 2xy)dx + (x^2 - 3y^2)dy$ *, where C is a curve defined by* $x = e^t \sin t, y = e^t \cos t, 0 \le t \le \pi$.

Solution: To find the line integral, look at the starting and ending point of the curve. When $t = 0, x = 0, y = 1$. When $t = \pi, x = 0, y = -e^{\pi}$. Hence

$$\int_C (3+2xy)dx+(x^2-3y^2)dy = \int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(\pi))-f(\vec{r}(0)) = f(0, -e^{\pi})-f(0, 1) = e^{3\pi}+1.$$

Example 79. *Given* $f(x, y, z) = xz + y^2z$ *is a potential function of the vector field* $\vec{F} = (z, 2yz, x+y^2)$. *Find* $\int_C \vec{F}\cdot d\vec{r}$ *, where C is a curve defined by* $x = t, y = t^2, z = 2t, 0 \le t \le 1$.

Solution: To find the line integral, look at the starting and ending point of the curve. When $t = 0, x = 0, y = 0, z = 0$. When $t = 1, x = 1, y = 1, z = 2$. Hence

$$\int_C \vec{F} \cdot d\vec{r} = f(1, 1, 2) - f(0, 0, 0) = 4.$$

A new method to find f such that $\vec{F}(x, y, z) = \nabla f$:

$$f(x, y, z) = \int_0^1 \vec{F}(tx, ty, tz) \cdot (x, y, z)\ dt + c.$$

Proof. Let C be the line segment from $P_0(x_0, y_0, z_0)$ to $P(x, y, z)$, where P_0 is arbitrary.
$C : \vec{r}(t) = (1 - t)P_0 + tP = ((1 - t)x_0 + tx, 1 - t)y_0 + ty, 1 - t)z_0 + tz), 0 \le t \le 1$.
Usually we take $P_0 = (0, 0, 0)$. Then
$C : \vec{r}(t) = tP = (tx, ty, tz), 0 \le t \le 1$.

$$f(x, y, z) = \int_C \vec{F}(x, y, z) \cdot d\vec{r} + c = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt + c$$
$$= \int_0^1 \vec{F}(tx, ty, tz) \cdot \vec{r}\,'(t)\ dt + c.$$

Example 80. *Find $f(x, y, z)$ such that $\nabla f = \vec{F} = (z, 2yz, x + y^2)$, and $f(1, 1, -1) = 5$.*

Solution: Take $P_0 = (0, 0, 0)$ (P_0 is always arbitrary). Let $C : \vec{r}(t) = (1 - t)(0, 0, 0) + t(x, y, z) = (tx, ty, tz), 0 \le t \le 1$.

$$f(x, y, z) = \int_C \vec{F} \cdot d\vec{r} + c$$
$$= \int_0^1 \vec{F}(tx, ty, tz) \cdot \vec{r}\,'(t)\ dt + c$$
$$= \int_0^1 \vec{F}(tx, ty, tz) \cdot (x, y, z)\ dt + c$$
$$= \int_0^1 (tz, 2tytz, tx + (ty)^2) \cdot (x, y, z)\ dt + c$$

$$= \int_0^1 (tzx + 2t^2y^2z + tzx + t^2y^2z)\ dt + c$$

$$= \int_0^1 (2tzx + 3t^2y^2z)\ dt + c$$

$$= xz + y^2z + c.$$

$$f(1, 1, -1) = 5, \Rightarrow -2 + c = 5, \Rightarrow c = 7.$$

$$f(x, y, z) = xz + y^2z + 7.$$

16.4 Green's theorem.

Green's theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C.

Green's Theorem. Let C be a positively oriented (counter-clockwise), piecewise smooth, simple closed curve in the plane, and let D be the region bounded by C. Let $\vec{F}(x,y) = (P(x,y), Q(x,y))$. If P and Q are functions of (x,y) defined on an open region containing D and have continuous partial derivatives there, then

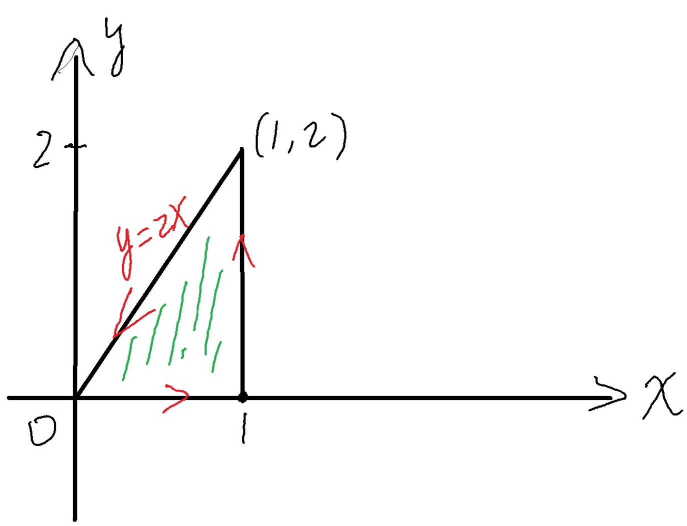
$$\oint_C P dx + Q dy = \oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

Properties:

- If $\vec{F}$ is conservative, then $\oint_C \vec{F} \cdot d\vec{r} = 0$.
- For a closed region R with boundary C, the area of R is:

$$A(R) = \frac{1}{2} \int_C x dy - y dx.$$

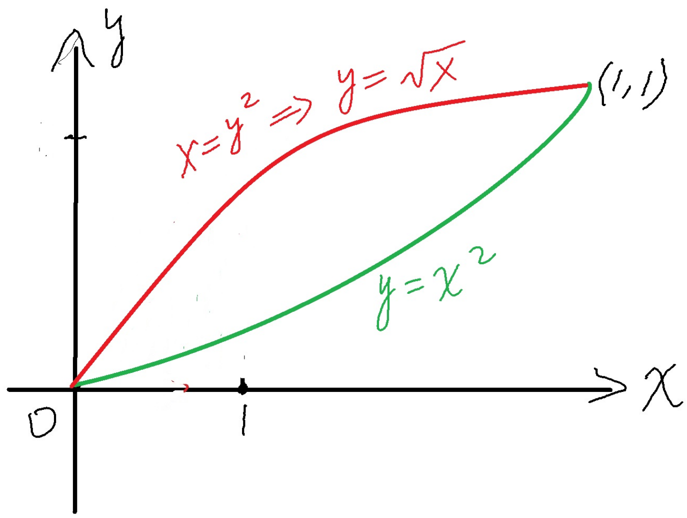
Example 81. *Use Green's Theorem to evaluate* $\oint_C xydx + x^2y^3dy$ *where* C *is the triangle with vertices* $(0,0), (1,0), (1,2)$*, with positive orientation.*



Solution: D is the triangle with vertices $(0,0),(1,0),(1,2)$. Then $D = R_{yx} : 0 \le y \le 2x, 0 \le x \le 1$.

$$\oint_C xydx + x^2y^3dy = \iint_D (2xy^3 - x)dxdy = \int_0^1 (\int_0^{2x} (2xy^3 - x)dy)dx = \frac{2}{3}.$$

Example 82. *Use Green's Theorem to evaluate* $\int_C (\sin x + y^3)dx + e^{y^2}dy$*, where* C *is the perimeter of the bounded region bounded by* $x = y^2$ *and* $y = x^2$ *with positive orientation.*



Solution: Note that $Q_x - P_y = -3y^2$. By Green's Theorem,

$$\int_C (\sin x + y^3)dx + e^{y^2}dy = \iint_D (-3y^2)dA.$$

To find D, the intersection between $x = y^2$ and $y = x^2$: (0,0), (1,1). Thus

$$D: \; x^2 \le y \le \sqrt{x}, \; 0 \le x \le 1.$$

Hence

$$\int_C (\sin x + y^3)dx + e^{y^2}dy = \iint_D (-3y^2)dA = \int_0^1 \int_{x^2}^{\sqrt{x}} (-3y^2)dydx$$

$$= \int_0^1 (-y^3)|_{y=x^2}^{\sqrt{x}} dx = \int_0^1 (-x^{3/2} + x^6)dx = (-\frac{2}{5}x^{5/2} + \frac{1}{7}x^7)|_0^1 = -\frac{9}{35}.$$

16.5 Curl and divergence.

Divergence measures the magnitude of a vector field's source or sink at a given point, in terms of a signed scalar. The curl of a vector field measures how a fluid may rotate.

Let

$$\vec{F} = (P(x,y,z), Q(x,y,z), R(x,y,z)),$$
$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}),$$

the vector differential operator.

- The divergence of $\vec{F}$ is: div $\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \vec{F}$.
- The curl of $\vec{F}$ is:

$$curl\vec{F} = \nabla\times\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

- If curl $\vec{F} = \vec{0}$ at a point P, then $\vec{F}$ is said to be irrotational at P. The $\vec{F}$ is conservative.
- $div\ curl\vec{F} = 0$.

Example 83. *Let $\vec{F}(x,y,z) = (xz, xy^3, xyz) = xz\vec{i} + xy^3\vec{j} + xyz\vec{k}$. Find div $\vec{F}(x,y,z)$, div $\vec{F}(1,-2,-1)$, curl $\vec{F}(x,y,z)$, curl $\vec{F}(0,1,1)$.*

Solution: Let $P(x,y,z) = xz, Q(x,y,z) = xy^3, R(x,y,z) = xyz$. Then
div $\vec{F}(x,y,z) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = z + 3xy^2 + xy$,
div $\vec{F}(1,-2,-1) = -1 + 12 - 2 = 9$,
curl $\vec{F}(x,y,z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xy^3 & xyz \end{vmatrix} = (xz, x - yz, y^3)$,
curl $\vec{F}(0,1,1) = (0,-1,-1)$.

16.6 Parametric surfaces and their areas.

Let $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$, $(u,v) \in D$. Then $\{(x,y,z) : x = x(u,v), y = y(u,v), z = z(u,v), (u,v) \in D\}$ is called a parametric surface S represented by $\vec{r}$.

Some special surfaces:

- Ellipsoid

  1. Cartesian (rectangular) equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
  2. Parametric form $x = a \sin\phi \cos\theta, y = b \sin\phi \sin\theta, z = c\cos\phi$, where $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.

- Elliptic cone

  1. Cartesian equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$.
  2. Parametric form $x = av\cos\theta, y = bv\sin\theta, z = cv$, where $0 \le \theta \le 2\pi$, $v \in \mathbb{R}$.

- Elliptic cylinder

  1. Cartesian equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
  2. Parametric form $x = a\cos\theta, y = b\sin\theta, z = v$, where $0 \le \theta \le 2\pi$.

- Hyperbolic cylinder

  1. Cartesian equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.
  2. Parametric form $x = a\sinh u, y = b\cosh u, z = v$, where $u, v \in \mathbb{R}$.

- Hyperbolic paraboloid

  1. Cartesian equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -z$.
  2. Parametric form $x = a\sqrt{v}\sinh u, y = b\sqrt{v}\cosh u, z = v$, where $u, v \in \mathbb{R}$.

- Torus

  1. Cartesian equation $x^2 + y^2 + z^2 + c^2 - a^2 - 2c\sqrt{x^2 + y^2} = 0$.
  2. Parametric form $x = (c + a\cos v)\cos u$, $y = (c + a\cos v)\sin u$, $z = a\sin v$, where $0 \le u, v < 2\pi$, $c > a > 0$.

Example 84. *Ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

can be parameterized as

$$x = a\sin\phi\cos\theta, y = b\sin\phi\sin\theta, z = c\cos\phi,$$

where $0 \le \theta \le 2\pi$*,* $0 \le \phi \le \pi$*.*

Tangent plane. The tangent plane of the parametric surface S at a point (u_0, v_0) is the plane containing the two tangent vectors $\vec{r}_u(u_0, v_0)$ and $\vec{r}_v(u_0, v_0)$.

Surface area.

Case 1: For a surface S: $z = f(x, y)$, $(x, y) \in D$, the area of the surface is

$$\iint_S dS = \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA.$$

$$dS = \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA.$$

Example 85. *Find the area of the surface* $z = 2y + \frac{2}{3}x^{3/2}$ *that lies directly above the region* $D = \{(x, y) : 4 \leq x \leq 11, 0 \leq y \leq 3\}$.

Solution:

$$Surface\ Area\ = \iint_D \sqrt{(z_x)^2 + (z_y)^2 + 1}\, dA$$

$$= \iint_D \sqrt{(\sqrt{x})^2 + 2^2 + 1}\, dA = \iint_D \sqrt{x+5}\, dA$$

$$\int_4^{11} \int_0^3 \sqrt{x+5} dy\ dx = \int_4^{11} 3\sqrt{x+5} dx$$

$$= 2(x+5)^{3/2}\Big|_4^{11} = 2(64 - 27) = 74.$$

Case 2: S: $\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$, $(u, v) \in D$. Assume that S is covered just once as (u, v) varies throughout D, then the surface area of S is

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| dA,$$

where

$$\vec{r}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}, \quad \vec{r}_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}.$$

Example 86. *Find the surface area of a sphere with radius* r.

Solution: The parametric surface is: $x = r \sin\phi \cos\theta, y = r \sin\phi \sin\theta, z = r\cos\phi$, where $(\phi, \theta) \in D = \{0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$.

$$\vec{r}_\phi \times \vec{r}_\theta = r^2(\sin^2\phi \cos\theta, \sin^2\phi \sin\theta, \sin\phi\cos\phi),$$

$$|\vec{r}_\phi \times \vec{r}_\theta| = r^2 \sin\phi.$$

$$A(S) = \iint_D |\vec{r}_\phi \times \vec{r}_\theta| dA = 4\pi r^2.$$

Example 87. *Find the area of the surface $z = x^2 + y^2$ that lies under the plane $z = 4$.*

Solution: Projecting the surface to xy-plane, we get $D: \ x^2 + y^2 \leq 4$.

$$A = \iint_D \sqrt{1 + z_x^2 + z_y^2} dA = \iint_D \sqrt{1 + 4(x^2 + y^2)} dA$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} \, r dr d\theta$$

$$= 2\pi \, \frac{1}{12} \left(1 + 4r^2\right)^{3/2} \Big|_0^1 = \frac{\pi}{6}(5\sqrt{5} - 1).$$

16.7 Surface Integrals

1. Surface Integrals of Scalar Fields

Let $f(x, y, z)$ be a function defined in a region in space containing a surface S. The surface integral of f over S is

$$\iint_S f(x, y, z)dS.$$

- If S is defined by $z = g(x, y)$ and the projection of S onto the xy-plane is D, then
$$\iint_S f(x, y, z)dS = \iint_D f(x, y, g(x, y))\sqrt{g_x^2 + g_y^2 + 1}dA.$$
- If S is defined by $y = g(x, z)$ and the projection of S onto the xz-plane is D, then
$$\iint_S f(x, y, z)dS = \iint_D f(x, g(x, z), z)\sqrt{g_x^2 + g_z^2 + 1}dA.$$
- If S is defined by $x = g(y, z)$ and the projection of S onto the yz-plane is D, then
$$\iint_S f(x, y, z)dS = \iint_D f(g(y, z), y, z)\sqrt{g_y^2 + g_z^2 + 1}dA.$$
- Parametric Surfaces: Let S be be a smooth surface with parametric representation $\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$, $(u, v) \in D$. Then
$$\iint_S f(x, y, z)dS = \iint_D f(\vec{r}(u, v))|\vec{r}_u \times \vec{r}_v|dA.$$

The unit normal vector of the surface S is

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}.$$

Example 88. *Evaluate $\iint_S ydS$, where S is the part of the plane $2x + 2y + z = 8$ that lies in the first octant .*

Solution: R is the triangle bounded by lines $x = 0$, $y = 0$, $2x + 2y = 8$. Note that

$$R = R_{yx} = \{(x, y) : 0 \le y \le 4 - x, 0 \le x \le 4\}.$$

By $z = 8 - 2x - 2y$, $z_x = -2$, $z_y = -2$. Thus

$$\iint_S zdS = \iint_R z\sqrt{z_x^2 + z_y^2 + 1}dA = \iint_R (8 - 2x - 2y)\sqrt{(-2)^2 + (-2)^2 + 1}dA$$

$$= 3\int_0^4 \int_0^{4-x} (8 - 2x - 2y)dydx = 3\int_0^4 (4 - x)^2dx = 64.$$

Example 89. *Evaluate* $\iint_S \frac{x+y}{\sqrt{2z+1}} dS$*, where* S *is the surface given by* $\vec{r}(u,v) = (u+v)\vec{i} + (u-v)\vec{j} + (u^2+v^2)\vec{k}$*,* $(u,v) \in D = \{0 \le u \le 1, 0 \le v \le 2\}$.

Solution: $\vec{r}_u \times \vec{r}_v = 2(u+v)\vec{i} + 2(u-v)\vec{j} - 2\vec{k}$. Thus

$$\iint_S \frac{x+y}{\sqrt{2z+1}} dS = 4\int_0^2 \int_0^1 u\, du dv = 4.$$

2. Surface Integrals of Vector Fields:

A surface S is orientable or two-sided if it has a unit normal vector $\vec{n}$ that varies continuously over S. Let $\vec{F}$ be a continuous vector field defined in a region containing an oriented surface S with unit normal vector $\vec{n}$. The surface integral (also called **flux integral**) of $\vec{F}$ across S in the direction of $\vec{n}$ is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS.$$

Remark: **flux** is the amount of "something" crossing a surface, such as water, wind, electric field.

If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, and S is given by $z = f(x,y)$ oriented upward and D is the projection onto the xy-plane, then

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D (-Pz_x - Qz_y + R) dA.$$

Example 90. *Find the flux of the vector field* $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$*,* S *is the surface that is composed of the part of the paraboloid* $z = \sqrt{4 - x^2 - y^2}$ *lying inside* $x^2 + y^2 = 1$.

Solution: Project S to xy-plane, we get $D: \ x^2 + y^2 \le 1$. Change D to $R = \{(r,\theta) : 0 \le r \le 1, 0 \le \theta \le 2\pi\}$.

$$flux = \iint_S \vec{F} \cdot d\vec{S} = \iint_D (-Pz_x - Qz_y + R) dA = \iint_D \frac{4}{\sqrt{4 - x^2 - y^2}} dA$$

$$= \iint_R \frac{4}{\sqrt{4 - r^2}} r dr d\theta.$$

Parametric Surfaces: Let S be be a smooth oriented surface with parametric representation $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$, $(u,v) \in D$.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) dA.$$

Example 91. *Find the flux of the vector field $\vec{F} = z\vec{i} + y\vec{j} + x\vec{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.*

Solution: $\vec{r}(\phi,\theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, where $(\phi,\theta) \in D = \{0 \le \phi \le \pi, 0 \le \theta \le 2\pi\}$.

$$\vec{r}_\phi \times \vec{r}_\theta = (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi),$$

$$\vec{F} \cdot (\vec{r}_\phi \times \vec{r}_\theta) = 2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta.$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) dA = \frac{4\pi}{3}.$$

16.8 Stokes' Theorem

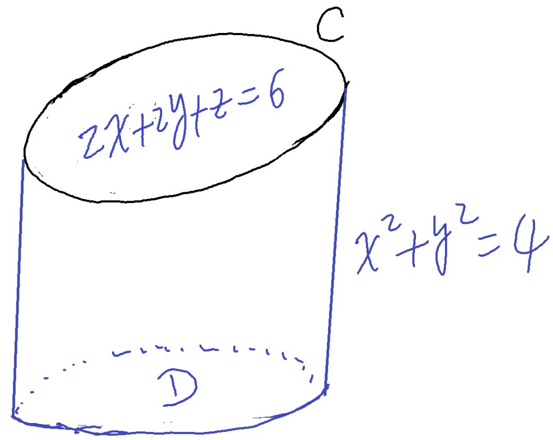
Stokes' theorem relates a line integral over a closed curve to a surface integral. It is a generalization of Green's Theorem to higher dimension.

Let S be an oriented piecewise-smooth surface that has a unit normal vector $\vec{n}$ and is bounded by a simple closed positively oriented curve C. If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field, where P, Q, R have continuous partial derivatives on an open region containing S, then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

Remark. $\nabla \times \vec{F} = curl\vec{F}$, $d\vec{S} = \vec{n}dS$.
Physical interpretation: If $\vec{F}$ is a force field, then the work done by $\vec{F}$ along C = the flux of curl$\vec{F}$ across S.

Example 92. *Let $\vec{F} = z\vec{i} + x^2\vec{j} + 2y\vec{k}$, and let S be the surface whose boundary C is the curve of intersection of the plane $2x + 2y + z = 6$ and the cylinder $x^2 + y^2 = 4$. Using Stokes' Thm evaluate $\int_C \vec{F} \cdot d\vec{r}$.*



Solution: By Stokes' Theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S}$$

$$= \iint_D (-Pz_x - Qz_y + R)dA,$$

where the surface S is $z = f(x,y) = 6 - 2x - 2y$, $(P,Q,R) = \nabla\times\vec{F}$.

By $z = 6 - 2x - 2y$, $z_x = -2$, $z_y = -2$.

$$\nabla\times\vec{F},\ or,\ curl\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = 2\vec{i} + \vec{j} + 2x\vec{k}.$$

Thus

$$(P,Q,R) = (2,1,2x).$$

The projection D of S on the xy-plane is the disk $x^2+y^2 \le 4$. By polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$,

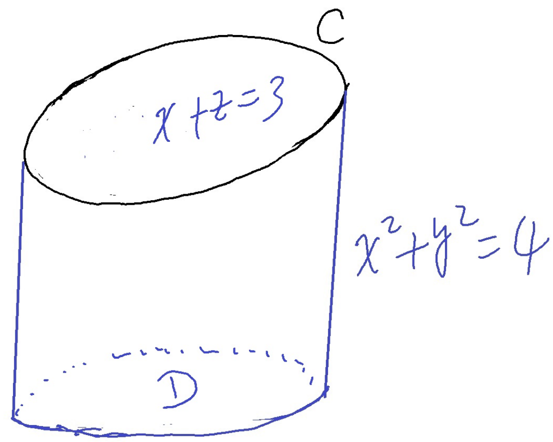
$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S}$$

$$= \iint_D (-Pz_x - Qz_y + R)dA$$

$$= \iint_D (6+2x)\ dA$$

$$= \int_0^2 \int_0^{2\pi} (6 + 2r\cos\theta) r dr d\theta = 24\pi.$$

Example 93. *Let $\vec{F} = (-3y^2, 2x, \sin(z^2+1))$, and let S be the surface whose boundary C is the curve of intersection of the plane $x + z = 3$ and the cylinder $x^2 + y^2 = 4$. Using Stokes' Thm evaluate $\int_C \vec{F} \cdot d\vec{r}$.*



Solution: By Stokes' Theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

$$= \iint_D (-Pz_x - Qz_y + R) dA,$$

where $\nabla \times \vec{F} = (P, Q, R)$. The surface S: $z = 3 - x$. Thus $z_x = -1$, $z_y = 0$.

$$\nabla \times \vec{F},\ or,\ curl\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = (2+6y)\vec{k}.$$

Thus

$$(P, Q, R) = (0, 0, 2 + 6y).$$

The projection D of S on the xy-plane is the disk $x^2 + y^2 \leq 4$. By polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

$$= \iint_D (-Pz_x - Qz_y + R) dA$$

$$= \iint_D (2 + 6y)\ dA$$

$$= \int_0^2 \int_0^{2\pi} (2 + 6r \sin\theta) r dr d\theta = 8\pi.$$

Example 94. *Let $\vec{F} = z\vec{i} + 6x\vec{j} + 2y\vec{k}$, and let S be the surface whose boundary C is the curve of intersection of the plane $2x + 2y + z = 6$ and the cylinder $x^2 + y^2 = 4$. Using Stokes' Thm find the flux of $\nabla \times \vec{F}$, i.e., evaluate $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$.*

Solution: By Stokes' Theorem,

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}.$$

For the curve C, let $x = 2\cos t$, $y = 2\sin t$, then $z = 6 - 4\cos t - 4\sin t$, $0 \le t \le 2\pi$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{r}\,'(t) dt$$

$$= \int_0^{2\pi} (6 - 4\cos t - 4\sin t, 12\cos t, 4\sin t) \cdot (-2\sin t, 2\cos t, 4\sin t - 4\cos t) dt$$

$$= \int_0^{2\pi} (-12\sin t - 8\sin t \cos t + 24) dt$$

$$= (12\cos t - 4\sin^2 t + 24t)|_{t=0}^{2\pi} = 48\pi$$

16.9 The Divergence Theorem

The divergence theorem relates a surface integral to a triple integral.

Let E be a simple solid region bounded by a closed piecewise-smooth surface S, and let $\vec{n}$ be the unit outer normal to S. If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field, where P, Q, R have continuous partial derivatives on an open region containing E, then

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS = \iiint_E div \vec{F} dV.$$

Example 95. *Evaluate $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F} = 3xz^2\vec{i} + 3y\vec{j} - z^3\vec{k}$, S is the unit sphere $x^2 + y^2 + z^2 = 1$.*

Solution:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E div\vec{F} dV = \iiint_E 3dV = 3 \cdot \frac{4}{3}\pi(1)^3 = 4\pi.$$

Example 96. *Evaluate* $\iint_S \vec{F} \cdot d\vec{S}$*, where* $\vec{F} = 3xz^2\vec{i} + (3y + yz)\vec{j} - z^3\vec{k}$*,* S *is the unit sphere* $x^2 + y^2 + z^2 = 1$*.*

Solution:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E div\vec{F} dV = \iiint_E (3 + z) dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 (3 + \rho\cos\phi)\ \rho^2 \sin\phi d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\pi} \left. (\rho^3 + \frac{1}{4}\rho^4 \cos\phi) \right|_{\rho=0}^{1} \sin\phi d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} (\sin\phi + \frac{1}{4}\cos\phi \sin\phi) d\phi d\theta = \int_0^{2\pi} (-\cos\phi + \frac{1}{8}\sin^2\phi)|_{\phi=0}^{\pi}\ d\theta$$

$$= \int_0^{2\pi} (2)\ d\theta = 4\pi.$$

Example 97. *Evaluate* $\iint_S \vec{F} \cdot d\vec{S}$*, where* $\vec{F} = 2xy^2\vec{i} + 2x^2y\vec{j} + z\vec{k}$*,* S *consists of three surfaces:* $z = 4 - 3x^2 - 3y^2$*,* $1 \le z \le 4$ *on the top;* $x^2 + y^2 = 1$*,* $0 \le z \le 1$ *on the side;* $z = 0$ *on the bottom.*

Solution:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E div\vec{F} dV = \iiint_E (2x^2 + 2y^2 + 1) dV.$$

$$= \int_0^{2\pi} \int_0^1 \int_0^{4-3r^2} (2r^2 + 1)\ r dz dr d\theta = 2\pi \int_0^1 (4 - 3r^2)(2r^3 + r)\ dr$$

$$= \frac{9\pi}{2}.$$