16. Integrals of Rational Functions & Partial Fractions

Lec 15 mini review.

useful trig identities:

$\cos^2(x) + \sin^2(x) = 1$

$1 + \tan^2(x) = \sec^2(x)$

$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

$\sin(2x) = 2\sin(x)\cos(x)$

expression:	**identity:**	**substitution:**
$\sqrt{1-x^2}$	$1 - \sin^2\theta = \cos^2\theta$	$x = \sin\theta$ $(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})$
$\sqrt{1+x^2}$	$1 + \tan^2\theta = \sec^2\theta$	$x = \tan\theta$ $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$
$\sqrt{x^2-1}$	$\sec^2\theta - 1 = \tan^2\theta$	$x = \sec\theta$ $(0 \leq \theta < \frac{\pi}{2},\quad \pi \leq \theta < \frac{3\pi}{2})$

STRATEGIES FOR INTEGRATING RATIONAL FUNCTIONS

Recall: a **rational function** is of the form $f(x) = \frac{N(x)}{D(x)}$ where the numerator $N(x)$ and the denominator $D(x)$ are both polynomials.

We already know how to integrate some rational functions:

$$\int \frac{1}{x}dx = \ln|x| + C$$

$$\int \frac{a}{bx+c}dx = \frac{a}{b}\ln|bx+c| + K$$

$u = bx + c$

$\frac{du}{dx} = b \Rightarrow dx = \frac{1}{b}du$

$$\int \frac{1}{x^2+1}dx = \arctan(x) + C$$

$$\int \frac{g'(x)}{g(x)}dx = \ln|g(x)| + C$$

$u = g(x)$

$\frac{du}{dx} = g'(x)$

$dx = \frac{du}{g'(x)}$

(where $g(x)$ is a polynomial)

Observation: the above forms of rational functions all have the property that the degree of the numerator is less than the degree of the denominator.

* These notes are solely for the personal use of students registered in MAT1320.

- Now, we consider a new way of expressing a rational function $\frac{N(x)}{D(x)}$ as a sum of simpler fractions.
- Before we can use this idea, we must, if necessary, reduce the integrand into a **proper** rational function, meaning one whose numerator $N(x)$ and denominator $D(x)$ satisfy

$$\deg(N) < \deg(D)$$

- If $\deg(N) \geq \deg(D)$, then $\frac{N(x)}{D(x)}$ is called an **improper rational function**.
- We can use long division to turn any improper rational function into one that is proper.

Example 16.1. $\displaystyle\int \frac{1}{x^2-1}dx$ $\quad \frac{1}{x^2-1}$ is a proper rational function

idea: rethink $\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)}$

Now "un-add" this fraction by deconstructing its (common) denominator

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

goal: solve for A and B so this equation is true

$$\Rightarrow \frac{1}{(x-1)(x+1)} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

$$\Rightarrow 1 = A(x+1)+B(x-1)$$

$$\Rightarrow 1 = Ax + A + Bx - B$$

$$\Rightarrow 1 = (A+B)x + (A-B)$$

L.S. $= 0x + 1$, R.S. $= (A+B)x + (A-B)$

$0 = A+B$ $\qquad 1 = A-B$

Now we have two equations with two unknowns:

① $A+B=0$

② $A-B=1$

①+②: $2A+B-B=0+1$

$\Rightarrow 2A = 1$

$\Rightarrow A = \frac{1}{2}$

From ① $B = -A = -\frac{1}{2}$

Let's check: $\frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)} = \frac{(x+1)-(x-1)}{2(x-1)(x+1)} = \frac{1}{x^2-1}$ ✓

Thus $\int \frac{1}{x^2-1}dx = \int \frac{\frac{1}{2}}{x-1}dx + \int \frac{-\frac{1}{2}}{x+1}dx$

$$= \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{2}\int \frac{1}{x+1}dx$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

Example 16.2. $\int \frac{2x+3}{x^2+5x+6}dx$ ← a proper rational function.

$$\frac{2x+3}{x^2+5x+6} = \frac{2x+3}{(x+3)(x+2)} = \frac{A}{x+3} + \frac{B}{x+2} = \frac{A(x+2)+B(x+3)}{(x+3)(x+2)}$$

$$\Rightarrow 2x+3 = A(x+2) + B(x+3)$$

time-saver: plug in $x=-2$: $2(-2)+3 = A(-2+2) + B(-2+3)$

$$\Rightarrow -1 = B$$

plug in $x=-3$: $2(-3)+3 = A(-3+2) + B(-3+3)$

$$\Rightarrow -3 = -A$$
$$\Rightarrow 3 = A$$

double check: $\frac{3}{x+3} - \frac{1}{x+2} = \frac{3(x+2)-(x+3)}{(x+3)(x+2)} = \frac{2x+3}{(x+3)(x+2)}$ ✓

Thus, $\int \frac{2x+3}{x^2+5x+6}dx = \int \left(\frac{3}{x+3} - \frac{1}{x+2}\right)dx$

$$= 3\int \frac{1}{x+3}dx - \int \frac{1}{x+2}dx$$

$$= 3\ln|x+3| - \ln|x+2| + C$$

Example 16.3. $\displaystyle\int \frac{2x^3 - 4x^2 + 10x + 1}{x^2 - 2x + 5}\,dx$ ← improper since deg. num > deg. denom

Long division:

$$\begin{array}{r} 2x \\ x^2-2x+5 \overline{\big)\, 2x^3-4x^2+10x+1} \\ -\underline{\left(2x^3-4x^2+10x\right)} \\ 0+1 \end{array}$$

$$\therefore \frac{2x^3-4x^2+10x+1}{x^2-2x+5} = 2x + \frac{1}{x^2-2x+5}$$

$$\Rightarrow \int \frac{2x^3-4x^2+10x+1}{x^2-2x+5}\,dx = \int 2x\,dx + \int \frac{1}{x^2-2x+5}\,dx$$

But x^2-2x+5 does not factor:

discriminant $b^2-4ac = 4-4(1)(5) < 0$

What to do?

Complete the square: $x^2-2x+5 = x^2-2x+1-1+5 = (x-1)^2+4$

then $\displaystyle\int \frac{1}{x^2-2x+5}\,dx = \int \frac{1}{(x-1)^2+4}\,dx$ ← this is an arctan in disguise

$$= \int \frac{1}{\frac{4}{4}(x-1)^2+4}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{x-1}{2}\right)^2+1}\,dx \qquad u = \frac{x-1}{2}, \quad \frac{du}{dx} = \frac{1}{2}$$

$$= \frac{1}{4}\int \frac{1}{u^2+1}(2du)$$

$$= \frac{1}{2}\arctan(u) + C$$

$$= \frac{1}{2}\arctan\left(\frac{x-1}{2}\right) + C$$

$$\Rightarrow \int \frac{2x^3-4x^2+10x+1}{x^2-2x+5}\,dx = \int 2x\,dx + \int \frac{1}{x^2-2x+5}\,dx$$

$$= x^2 + \frac{1}{2}\arctan\left(\frac{x-1}{2}\right) + C$$

Partial Fractions with Repeated Factors

Once you have used long division to obtain a proper rational function, you need to factor its denominator $D(x)$.

Every polynomial can be factored into a product of linear factors (of the form $ax + b$) and irreducible quadratic factors (of the form $ax^2 + bx + c$ where $b^2 - 4ac < 0$)

- For each distinct linear factor of the denominator $D(x)$ – which may be a repeated factor (say, to the power r)
$$(ax + b)^r$$
the partial fractions decomposition will have r terms corresponding to the factor $(ax + b)^r$:
$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r}$$
(ascending powers of ax+b in denominators)

- For each distinct irreducible quadratic factor of the denominator $D(x)$ – which may be a repeated factor (say, to the power r)
$$(ax^2 + bx + c)^r \quad \text{(where } b^2 - 4ac < 0\text{)}$$
the partial fractions decomposition will have r terms corresponding to the factor $(ax^2 + bx + c)^r$:
$$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$
(ascending powers of ax^2+bx+c in denominators)

Example 16.4. $\displaystyle\int \frac{2x^2 + 3x + 1}{(x + 2)(x - 5)^3(x^2 + 1)^2(x^2 - 6x + 13)}\,dx$

$(x+2)$: linear factor repeats 1 time

$(x-5)^3$: linear factor repeats 3 times

$(x^2+1)^2$: irreducible quadratic facto repeated 2 times

$(x^2-6x+13)$: irreducible quadratic factor repeats 1 time

this integrand has the following partial fractions decomposition

$$\therefore \frac{2x^2+3x+1}{(x+2)(x-5)^3(x^2+1)^2(x^2-6x+13)} = \frac{A}{x+2} + \frac{B}{x-5} + \frac{C}{(x-5)^2} + \frac{D}{(x-5)^3} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2-6x+13)}$$

Now we need to solve for the constants A, B, C, D, E, F, G, H, I, J yikes!

$$\therefore \frac{2x^2+3x+1}{(x+2)(x-5)^3(x^2+1)^2(x^2-6x+13)} = \frac{A}{x+2} + \frac{B}{x-5} + \frac{C}{(x-5)^2} + \frac{D}{(x-5)^3} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2-6x+13)}$$

In principle, this amounts to 10 equations in 10 unknowns...
Let's imagine we did this bit of algebra.
Then what remains is evaluating these partial fraction integrals:

$$\int \frac{A}{x+2}dx = A\ln|x+2| + K. \qquad \int \frac{B}{x-5}dx = B\ln|x-5| + K$$

$$\int \frac{C}{(x-5)^2}dx = -C(x-2)^{-1} + K \qquad \int \frac{D}{(x-5)^3}dx = -\frac{D}{2}(x-5)^{-2} + K$$

$$\int \frac{Ex+F}{x^2+1}dx = E\underbrace{\int \frac{x}{x^2+1}dx}_{\substack{u\text{-sub.} \\ u=x^2+1}} + F\int \frac{1}{x^2+1}dx = \frac{E}{2}\ln|x^2+1| + F\arctan(x) + K$$

$$\int \frac{Gx+J}{(x^2+1)^2}dx = G\underbrace{\int x(x^2+1)^{-2}dx}_{\substack{u\text{-sub.} \\ u=x^2+1}} + J\underbrace{\int \frac{1}{(x^2+1)^2}dx}_{\substack{\text{trig.sub.} \\ x=\tan\theta}} = -\frac{G}{2}(x^2+1)^{-1} + \frac{J}{2}\arctan(x) + \frac{J}{2}\left(\frac{x}{1+x^2}\right) + K$$

$$\int \frac{Ix+J}{x^2-6x+13}dx = \frac{1}{4}\int \frac{Ix+J}{\left(\frac{x-3}{2}\right)^2+1}dx = \frac{1}{4}\underbrace{\int \frac{Ix-3I}{\left(\frac{x-3}{2}\right)^2+1}dx}_{\substack{u\text{-sub.} \\ u=\left(\frac{x-3}{2}\right)^2+1}} + \frac{1}{4}\underbrace{\int \frac{3I+J}{\left(\frac{x-3}{2}\right)^2+1}dx}_{\substack{u\text{-sub.} \\ u=\frac{x-3}{2}}} = \frac{I}{4}\ln\left|\left(\frac{x-3}{2}\right)^2+1\right| + \frac{3I+J}{4}\left(2\arctan\left(\frac{x-3}{2}\right)\right) + K$$

Example 16.5. $\displaystyle\int \frac{x^2+2x+1}{(x^2+1)^2}dx$

x^2+1 is an irreducible quadratic factor since $b^2-4ac = 0^2-4(1)(1) < 0$

$$\frac{x^2+2x+1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

repeats twice

$$\Longrightarrow x^2+2x+1 = (Ax+B)(x^2+1) + Cx+D = Ax^3+Bx^2+(A+C)x + B+D$$

$$LS = 0x^3 + 1x^2 + 2x + 1 \qquad RS = Ax^3 + Bx^2 + (A+C)x + (B+D)$$

$$\therefore A=0,\ B=1,\ A+C=2,\ B+D=1$$
$$\Rightarrow C=2 \quad \Rightarrow D=0$$

$$\int \frac{x^2+2x+1}{(x^2+1)^2}dx = \int \frac{1}{x^2+1}dx + \int \frac{2x}{(x^2+1)^2}dx = \arctan(x) - (x^2+1)^{-1} + C \leftarrow \text{check!}$$

Example 16.6. $\int_2^3 \frac{2x+1}{x(x-1)^2}dx$

$$\frac{2x+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

repeats twice

$$\Rightarrow 2x+1 = A(x-1)^2 + Bx(x-1) + Cx$$

plug in $x=1$:
$\Rightarrow 2(1)+1 = A(0^2)+B(1)(0)+C(1)$
$\Rightarrow C=3$

plug in $x=0$:
$\Rightarrow 2(0)+1 = A(-1)^2+B(0)(-1)+C(0)$
$\Rightarrow A=1$

plug in $A=1, C=3, x=-1$:
$\Rightarrow 2(-1)+1 = (1)(-2)^2+B(-1)(-2)-3$
$\Rightarrow B=-1$

$$\therefore \int_2^3 \frac{2x+1}{x(x-1)^2}dx = \int_2^3\left(\frac{1}{x} - \frac{1}{x-1} + \frac{3}{(x-1)^2}\right)dx$$

$$= \left[\int\frac{1}{x}dx - \int\frac{1}{x-1}dx + 3\int\frac{1}{(x-1)^2}dx\right]_2^3$$

$$= \left[\ln|x| - \ln|x-1| - 3(x-1)^{-1}\right]_2^3$$

$$= \left(\ln(3) - \ln(3-1) - 3(3-1)^{-1}\right) - \left(\ln(2) - \ln(2-1) - 3(2-1)^{-1}\right)$$

$$= \ln(3) - \ln(2) - \frac{3}{2} - \ln(2) + \ln(1) + 3$$

$$= \ln(3) - 2\ln(2) + \frac{3}{2}$$

Study Guide

- ◇ **integrating rational functions:** $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|$ $\quad \int \frac{1}{x^2+1}dx = \tan^{-1}(x)$
- ◇ use long division to obtain a proper fraction
- ◇ factor denominator into product of linear and irreducible quadratic factors
- ◇ decompose integrand into its partial fractions

Exercises (Stewart, 8th ed.) §7.4, pg. 501 # 7, 9, 11, 15, 19, 21, 23, 27, 29, 39, 41, 43, 47