MATH 1007 B, Tutorial 6

Instructions: Work in teams of 3 or 4. At the end of the tutorial, each team hands in one set of solutions with everybody's name and student number. Do not divide up problems and work on them separately.

Question 1. [10 marks] Use the following properties of the function $f(x)$ defined $(-\infty, \infty)$ to sketch its graph. Identify (if they exist) the local min(s), max(s), inflection point(s) and vertical and horizontal asymptote(s).

(i) $f'(-1) = f'(1) = 0$ and $f(-1) = 3$ and $f(1) = 0$, where $f'(x)$ is the derivative of $f(x)$
(ii) $f''(-1) = -1$. And $f'(x) < 0$ on $(-1, 1)$ and $f'(x) > 0$ on $(1, \infty)$,
where $f''(x)$ is the second derivative of $f(x)$.
(iii) $f(0) = 1$
(iv) On $(-\infty, 1)$, $f''(x) < 0$. Also, on $x = 2$, f'' changes from positive to negative and $f(2) = \frac{1}{2}$.
(v) $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to+\infty} f(x) = 1$

Solution: See the attached paper

Question 2. Use the L'Hôpital's rule to evaluate the following limits when it is needed!

(a) [2] $\lim\limits_{x\to\frac{\pi}{4}+} \dfrac{\sin 2x}{x - \frac{\pi}{4}}$

Solution: $\lim\limits_{x\to\frac{\pi}{4}+} \dfrac{\sin 2x}{x - \frac{\pi}{4}} = \dfrac{\sin\frac{\pi}{2}}{0} = \infty$. (LHôpital's rule is not needed here)

(b) [3] $\lim\limits_{x\to\frac{\pi}{4}+} \dfrac{\cos 2x}{x - \frac{\pi}{4}}$

Solution: Using L'Hôpital's rule we have:
$\lim\limits_{x\to\frac{\pi}{4}+} \dfrac{\cos 2x}{x - \frac{\pi}{4}} = \lim\limits_{x\to\frac{\pi}{4}+} \dfrac{-2\sin 2x}{1} = -2$

(c) [5] $\lim\limits_{x\to\infty} x^3 e^{-x}$

Solution:

$$\begin{aligned}
\lim_{x\to\infty} x^3 e^{-x} &= \lim_{x\to\infty} \frac{x^3}{e^x} \\
&= \lim_{x\to\infty} \frac{3x^2}{e^x} \ (first\ application\ of\ L'Hôpital's\ rule) \\
&= \lim_{x\to\infty} \frac{6x}{e^x} \ (second\ application\ of\ L'Hôpital's\ rule)
\end{aligned}$$

$$= \lim_{x\to\infty} \frac{6}{e^x} \ (\textit{third application of } L'H\hat{o}pital's \textit{ rule})$$
$$= 0.$$

Question 3. In the following (a)-(c) find the function $f(x)$ when:

(a) [4] $f'(x) = x^{-5}$ and $f(1) = -\frac{1}{4}$

Solution: $f(x) = -\frac{1}{4x^4} + C$. $f(1) = -\frac{1}{4} + C = -\frac{1}{4} \to C = 0$. Hence, $f(x) = -\frac{1}{4x^4}$

(b) [5] $f'(x) = \dfrac{2}{1+x}$ and $f(e-1) = 0$

Solution: $f(x) = 2\ln|x+1| + C$. $f(e-1) = 2\ln e + C = 0 \to C = -2$. Hence, $f(x) = 2\ln|x+1| - 2$

(c) [6] $f'(x) = \dfrac{2x^3 + x}{x^2}$ and $f(1) = 1$

Solution: $f'(x) = 2x + \frac{1}{x}$ and as a result $f(x) = x^2 + \ln|x| + C$. $f(1) = 1^2 + 0 + C = 1 \to C = -1$. Hence, $f(x) = x^2 + \ln|x| - 1$.