University of Ottawa
Department of Mathematics and Statistics

MAT1322 Calculus II

Midterm test 3

November 25, 2019

Instructor: Vadim Kaimanovich

Duration: 75 minutes

Read the following information before starting the test:

- Verify that your copy of the test contains 5 pages, including this one.
- Write your name and student number on this page.
- Work the problems in the space provided. Use the back-pages and the blank sheet attached at the end for rough work. Do not use any other paper. Before submitting the test remove the rough work page 5.
- Show all work, clearly and in order, if you want to get full credit. Points may be taken off if it is not clear how you arrived at your answer (even if your final answer is correct).
- Please keep your written answers brief; be clear and to the point. Points may be taken off for rambling and for incorrect or irrelevant statements.
- Circle your final answers and enter them in the corresponding boxes at the bottom of this page.

The Faculty of Science requires that you read and sign the following statement: Cellular phones, calculators or other electronic devices and course notes are not allowed during this test. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the test.

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Problem	1(2)	2(2)	3(2)	4(2)	5(2)	6(2)	Total(12)
Answer	F	B	C	A	B	H	
Points							

ID checked:

1. Find the convergence radius R and the convergence interval I of the power series $\sum_{n=0}^{\infty} \frac{(3x+2)^n}{2^n \ln(n+2)}$.

A. $R = 0, I = [0, 4/3]$ B. $R = 3/2, I = [-3, 0)$ C. $R = 1, I = (-2/3, 2/3]$ D. $R = -2/3, I = (-4/3, 0)$
E. $R = 2/3, I = (-4/3, 0]$ F. $R = 2/3, I = [-4/3, 0)$ G. $R = 3/2, I = [-4/3, 0)$ H. none of the above

Let $a_n = \frac{(3x+2)^n}{2^n \ln(n+2)}$, then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3x+2}{2}\right| \frac{\ln(n+2)}{\ln(n+3)} \to \left|\frac{3x+2}{2}\right|$, so that by the ratio test the series converges for $|3x+2| < 2$ and diverges for $|3x+2| > 2$. Thus, the endpoints of the convergence interval are $x_1 = -\frac{4}{3}$, $x_2 = 0$, and $R = |x_2 - x_1|/2 = \frac{2}{3}$. It remains to investigate the behaviour of the series at the endpoints. For $x = -\frac{4}{3}$, $a_n = \frac{(-1)^n}{\ln(n+2)}$, so that the series converges by the alternating test. For $x = 0$, $a_n = \frac{1}{\ln(n+2)}$, and the series diverges.

2. By using differentiation or integration of power series find the power series representation $\sum_{n=0}^{\infty} a_n x^n$ of the function $f(x) = 1/(2-x)^2$ at the point 0. What are its convergence interval I and the coefficient a_2?

A. $I = (-1, 1), a_2 = 3$ B. $I = (-2, 2), a_2 = 3/16$ C. $I = (-1, 1), a_2 = 3/4$ D. $I = [-1, 1], a_2 = 3/8$
E. $I = [-2, 2], a_2 = 3/16$ F. $I = (-2, 2), a_2 = 3/8$ G. $I = (-2, 2), a_2 = 3/4$ H. none of the above

$f(x) = \frac{1}{(2-x)^2} = \left(\frac{1}{2-x}\right)'$, on the other hand,

$$\frac{1}{2-x} = \frac{1}{2} \cdot \frac{1}{1-\frac{x}{2}} = \frac{1}{2}\left(1 + \frac{x}{2} + \frac{x^2}{4} + \ldots\right) = \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \ldots$$

on the interior $\{x : |\frac{x}{2}| < 1\} = (-2, 2)$ of its interval of convergence. Thus, by differentiating the above series,

$f(x) = \frac{1}{4} + \frac{2}{8}x + \frac{3}{16}x^2 + \frac{4}{32}x^3 + \ldots$ with the same radius of convergence as the original series.

For $x = \pm 2$ $|a_n| \to \infty$, whence the interval of the power series representation of f is $(-2, 2)$.

3. What is the coefficient a_3 of the Taylor series $\sum_{n=0}^{\infty} a_n x^n$ of the function $f(x) = \sqrt{1-x}$ at the point 0? By using the Taylor series find the limit $L = \lim_{x\to 0}(\sqrt{1-x} - 1 + x/2)/x^2$.

A. $a_3 = 1/8, L = -1/8$ B. $a_3 = 1/2, L = 1/16$ C. $a_3 = -1/16, L = -1/8$ D. $a_3 = 1/8, L = 1/16$
E. $a_3 = 1/8, L = -1/4$ F. $a_3 = 1/4, L = -1/4$ G. $a_3 = -1/8, L = -1/16$ H. none of the above

$f(x) = (1-x)^{1/2}$, whence $f'(x) = -\frac{1}{2}(1-x)^{-1/2}$, $f''(x) = -\frac{1}{4}(1-x)^{-3/2}$,
$f'''(x) = -\frac{3}{8}(1-x)^{-\frac{5}{2}}$ and $f(0) = 1$, $f'(0) = -\frac{1}{2}$, $f''(0) = -\frac{1}{4}$,
$f'''(0) = -\frac{3}{8}$, whence the Taylor series is
$\sqrt{1-x} \sim 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \ldots$, so that $a_3 = -\frac{1}{16}$, and
$(\sqrt{1-x} - 1 + \frac{x}{2})/x^2 \sim -\frac{1}{8} - \frac{1}{16}x - \ldots$, so that $L = -\frac{1}{8}$

4. By using the method of separable variables find the solution of the differential equation $y' = -2xe^{-y}$ that satisfies the initial condition $y(-2) = 0$.

A. $y = \ln(5 - x^2)$ B. $y = \ln(x^2 - 5)$ C. $y = e^{1-x^2}$ D. $y = 1/x$ E. $y = -2\ln(x^2+1)$ F. $y = e^{x/2}$
G. $y = -\ln(1 - x^2/2)$ H. $y = \ln(x+3)$ I. $y = \ln(x^2 - 3)$ J. none of the above

$\frac{dy}{dx} = -2xe^{-y} \iff e^y dy = -2x\,dx \iff \int e^y dy = -\int 2x\,dx + C$
$\iff e^y = -x^2 + C \iff y = \ln(-x^2 + C)$
Then $y(-2) = \ln(-(-2)^2 + C) = \ln(-4 + C) = 0$, whence
$-4 + C = 1 \Rightarrow C = 5$

5. Which of the following functions $f = f(x, y)$ have the property that $f_{xx} + f_{yy} = 0$?
(I) $e^{-x^2+y^2}$, (II) $3x^2y + y^3$, (III) $\ln(x^2 + y^2)$, (IV) $e^x \sin y - e^{-y} \cos x$?

A. (II) and (IV) B. (III) and (IV) C. (I), (II) and (IV) D. (II), (III) and (IV) E. (I) and (III)
F. none G. all H. none of the above

(I) $\left(e^{-x^2+y^2}\right)''_{xx} = \left(-2xe^{-x^2+y^2}\right)'_x = -2e^{-x^2+y^2} + 4x^2e^{-x^2+y^2}$

$\left(e^{-x^2+y^2}\right)''_{yy} = \left(2ye^{-x^2+y^2}\right)'_y = 2e^{-x^2+y^2} + 4y^2e^{-x^2+y^2}$

(II) $\left(3x^2y + y^3\right)''_{xx} = 6y$, $\left(3x^2y + y^3\right)''_{yy} = 6y$

(III) $\left(\ln(x^2+y^2)\right)''_{xx} = \left(\frac{2x}{x^2+y^2}\right)'_x = \frac{2(x^2+y^2) - 2x \cdot 2x}{(x^2+y^2)^2} = \frac{-2x^2+2y^2}{(x^2+y^2)^2}$

in the same way by symmetry $\left(\ln(x^2+y^2)\right)''_{yy} = \frac{-2y^2+2x^2}{(x^2+y^2)^2}$

(IV) $\left(e^x \sin y - e^{-y}\cos x\right)''_{xx} = (e^x)''_{xx} \sin y - e^{-y}(\cos x)''_{xx} = e^x \sin y + e^{-y}\cos x$

$\left(e^x \sin y - e^{-y}\cos x\right)''_{yy} = e^x(\sin y)''_{yy} - (e^{-y})''_{yy}\cos x = -e^x \sin y - e^{-y}\cos x$

6. By using the chain rule find the vector of partial derivatives $\left(\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}, \frac{\partial z}{\partial u}\right)$ for $z = x^2 - xy^2$ with $x = 2s + t - u$ and $y = st^2u^2$ at the point $(s, t, u) = (2, -1, 1)$.

A. 0 B. $(0, -8)$ C. $(1, 4, -4)$ D. $2x - 2xy$ E. $(2s - t + u)^2 - (2s - t + u)(st^2u^2)^2$ F. $(-8, -32, 32)$
G. $(2, -1, 1)$ H. $(-8, 32, -32)$ I. $(1, -4, 4)$ J. none of the above

$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$; $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$; $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$;

The partial derivatives of x and y at a general point (s, t, u) are

$\frac{\partial x}{\partial s} = 2$, $\frac{\partial x}{\partial t} = 1$, $\frac{\partial x}{\partial u} = -1$, $\frac{\partial y}{\partial s} = t^2u^2$, $\frac{\partial y}{\partial t} = 2stu^2$; $\frac{\partial y}{\partial u} = 2st^2u$;

so that for $(s, t, u) = (2, -1, 1)$, $\frac{\partial y}{\partial s} = 1$, $\frac{\partial y}{\partial t} = -4$, $\frac{\partial y}{\partial u} = 4$

Further, $\frac{\partial z}{\partial x}(x, y) = 2x - y^2$, $\frac{\partial z}{\partial y}(x, y) = -2xy$, $x(2, -1, 1) = 2$, $y(2, -1, 1) = 2$,

and $\frac{\partial z}{\partial x}(2, 2) = 0$, $\frac{\partial z}{\partial y}(2, 2) = -8$. Thus,

$\frac{\partial z}{\partial s}(2, -1, 1) = -8$, $\frac{\partial z}{\partial t}(2, -1, 1) = 32$, $\frac{\partial z}{\partial u}(2, -1, 1) = -32$

extra page for calculations (please remove it when submitting the test!)