MATH 1007 B, Tutorial 4

Instructions: Work in teams of 3 or 4. At the end of the tutorial, each team hands in one set of solutions with everybody's name and student number. Do not divide up problems and work on them separately.

Question 1. Use the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ to answer the following Parts (a) and (b).

(a) [3] Evaluate $\lim_{x\to 0} \Big(\frac{\sin 3x}{6x}\Big)^2$.

Solution:

$$\lim_{x\to 0} \Big(\frac{\sin 3x}{6x}\Big)^2 = \lim_{x\to 0} \Big(\frac{1}{2} . \frac{\sin 3x}{3x}\Big)^2 = \frac{1}{4}\Big(\lim_{x\to 0} \frac{\sin 3x}{3x}\Big)^2 = \frac{1}{4}.1^2 = \frac{1}{4}.$$

(b) [4] Evaluate $\lim_{x\to 0} \frac{\cos x - 1}{x}$.

Hint: Recall the formula $2\sin^2 \frac{x}{2} = 1 - \cos x$.

Solution: $\lim_{x\to 0} \frac{-2\sin^2 \frac{x}{2}}{x} = -\lim_{x\to 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} . \lim_{x\to 0} \sin \frac{x}{2} = -1 \times 0 = 0$

Question 2. (a) [4] Consider the function

$$f(x) = \begin{cases} 2x^2 - 1, & x \geq 2; \\ \dfrac{5x}{x-4} + 12, & x < 2. \end{cases}$$

Is it continuous at $x = 2$? Why?

Solution: Yes. since, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2x^2 - 1 = 7$.

On the other hand $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{5x}{x-4} + 12 = 7$. And $f(2) = 2.2^2 - 1 = 7$.

Question 3. [6] Is $f(x) = \sqrt{x-1}$ continuous at $x = 1$? Why?

No. Since, $\lim_{x\to 0^+} \sqrt{x-1} = 0$ BUT $\lim_{x\to 0^-} \sqrt{x-1}$ does not exists.

Question 4. (a) [4] Show that the function $f(x) = \frac{x-1}{x+1} - 4 + x^2$ has a root in $[1, 2]$?

Solution: First observe that the function f is **continuous** on the interval $[1, 2]$. Now look at $f(1)$ and $f(2)$ which are, $f(1) = -3 < 0$ and $f(2) = \frac{1}{3} > 0$. Since, the continuous function f changes sign on the interval $[1, 2]$ it means that it intercepts the $x-$axis somewhere between 1 and 2.

(b) [4] Can you conclude that function $f(x) = \dfrac{x-3}{x+1} + x^2$ has a root in $[-2, 1]$? Why?

Solution: NO we can not make such a conclusion although the function f changes sign over the interval $[-2, 1]$. BUT, at $x = -1$ which is inside the interval $[-2, 1]$ this function is NOT CONTINUOUS. So we can not use the intermediate value theorem here.