

MAT2355 – INTRODUCTION TO GEOMETRY – SECOND ASSIGNMENT
 PROFESSOR : MAIA FRASER

Due date: February 12th (Submit a single PDF file on Brightspace by 3pm)

Name	Student Number
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*** You must upload a single PDF of this document with solutions added. Two ways to accomplish this: print the assignment, write your solutions in the designated spaces, and **scan** your solution to PDF; alternatively add your solutions to the given LaTeX template and compile to PDF.

*** Show your work. Justify your answers. Writing the final answer is not sufficient.

Problem	1	2	3	4	5	Total out of 19
Grade						

QUESTION 1. (a) (3 points) Compute a formula for the reflection $R : \mathbb{E}^3 \rightarrow \mathbb{E}^3$ by the plane $x + 2y = 1 - z$.

Your answer should determine the coordinates explicitly, i.e., it should be in the form $R(x, y, z) = (\dots, \dots, \dots)$ for every point $(x, y, z) \in \mathbb{E}^3$.

Solution: We can rewrite $x + 2y = 1 - z$ as $x + 2y + z = 1$ which corresponds to the plane $H = \{\vec{x} \in \mathbb{R}^3 : \vec{a} \cdot \vec{x} = b\}$ where $\vec{a} = (1, 2, 1)$ and $b = 1$. We find $\|\vec{a}\|^2 = 1 + (2)^2 + 1 = 6$ and so by the general formula for a reflection (see notes) the reflection is given by

$$\begin{aligned}
 R(x, y, z) &= (x, y, z) + 2 \frac{(b - \vec{a} \cdot (x, y, z))}{\|\vec{a}\|^2} \vec{a} \\
 &= (x, y, z) + \frac{2(1 - x - 2y - z)}{6} (1, 2, 1) \\
 &= (x, y, z) + \left(\frac{1}{3}(1 - x - 2y - z), \frac{1}{3}(2 - 2x - 4y - 2z), \frac{1}{3}(1 - x - 2y - z)\right) \\
 &= \left(\frac{1}{3}(1 + 2x - 2y - z), \frac{1}{3}(2 - 2x - y - 2z), \frac{1}{3}(1 - x - 2y + 2z)\right)
 \end{aligned}$$

Grading Scheme:

- 1 mark for (showing that you know) the description of H .
- 1 mark for (showing that you know) the correct formula for a reflection in a plane.
- 1 mark for the correct final answer.

- Remark: In all of the problems, for each part the points can be 0, $\frac{1}{2}$ or 1.

(b) (2 points) Consider the map $T : \mathbb{E}^3 \rightarrow \mathbb{E}^3$ given by $T(x, y, z) = (x, z, y)$. Find the equation of a plane $H \subset \mathbb{E}^3$ such that T is a reflection by H . You should justify that T is indeed a reflection by your plane H .

Hint: To find the plane, use the fact that it is fixed by T .

Solution: By the definition of a reflection through a plane H , it moves every point P by twice the distance between P and H . Therefore the set of fixed points of the reflection (i.e. points moved a distance zero) is H . But fixed points of T are the solutions to $T(\vec{v}) = \vec{v}$. If $\vec{v} = (x, y, z)$ then

$$T(\vec{v}) = \vec{v} \Rightarrow (x, z, y) = (x, y, z) \Rightarrow z = y.$$

Therefore the equation of the fixed point set of T is $y - z = 0$.

To prove T is in fact a reflection by this plane we need to use either the definition of a reflection (that we used in class to come up with the general formula for reflections) or the general formula itself. Here is an example of the latter approach:

We know the formula for a reflection through a plane $H = \{\vec{x} \in \mathbb{R}^3 : \vec{a} \cdot \vec{x} = b\}$, so we first write the plane $y - z = 0$ mentioned above as $\vec{a} \cdot \vec{x} = b$, by taking $\vec{a} = (0, 1, -1)$ and $b = 0$. Here $\|\vec{a}\|^2 = 2$, and so the general formula of a reflection through this plane is

$$R(x, y, z) = (x, y, z) + 2 \frac{(0 - (y - z))}{2} ((0, 1, -1)) = (x, y, z) + (0, z - y, y - z) = (x, z, y).$$

This is the same as the formula for T so we conclude $T = R$ and thus T is a reflection through the plane $y - z = 0$.

Grading Scheme:

- 1 mark for computing the equation of the plane which is the fixed point set of T .
- 1 mark for any reasonable justification that T is a reflection through this plane.
- Remark: Some students might solve this question in creative ways. Whatever argument is given, however, must be rigorous for full points. One alternative solution that some may attempt is to equate the general formula for a reflection with the given formula for T (namely $T(x, y, z) = (x, z, y)$) and then solve. Beware: this is much more work and this lengthy work increases the risk of making mistakes (ignoring hints is usually a bad idea!). If the argument is done completely correctly then it receives full points, however, mistakes or logical gaps along the way will result in the loss of points.

(c) (4 points) Suppose $S : \mathbb{E}^n \rightarrow \mathbb{E}^n$ is an isometry that fixes all the points of some hyperplane H in \mathbb{E}^n . Let P be an arbitrary point of \mathbb{E}^n and let Q be its orthogonal projection to H . Show that S maps P to a point P' such that $d(P, Q) = d(P', Q)$, and show also that P' belongs to the line through P and Q (Hint: consider the projection of P' to H). Can you conclude that S is a reflection? Explain.

Solution: Let $P \in \mathbb{E}^n$ be arbitrary and write $P' = S(P)$ and let Q be the orthogonal projection of P to H . We need to address three things: (1) show P' satisfies $d(P, Q) = d(P', Q)$, (2) show P' belongs to the line through P and Q , and (3) say whether S is reflection and justify.

(1) We have

$$\begin{aligned} d(P, Q) &= d(S(P), S(Q)) \text{ since } S \text{ is an isometry} \\ &= d(P', Q) \text{ by definition of } P' \text{ and using } Q \in H \end{aligned}$$

(2) Let M be the orthogonal projection of P' to H . The lines PQ and $P'M$ are orthogonal to H by definition of Q and M . So, if we show $M = Q$ we will know these lines coincide and it will follow that P' belongs to the line PQ . To establish $M = Q$ we recall that orthogonal projection has a distance-minimizing property. More specifically, the orthogonal projection Q of P has the property it is the point of H closest to P . Because S is an isometry that fixes all points of H , $S(Q)$ is therefore the point of H closest to $S(P)$. But $S(Q) = Q$ and $S(P) = P'$, so we have that Q is the closest point of H to P' . We now once again use the distance-minimizing property of orthogonal projection and conclude that Q is the orthogonal projection of P' to H , i.e. $Q = M$.

(3) *Aside: We know that P' belongs to the line PQ and is at a distance $d(P, Q)$ from $Q \in H$. There are two such points: P itself, and its reflection through H . This is the intuition that tells us S is not necessarily a reflection. The rigorous proof is as follows...*

We cannot conclude S is a reflection through H because the identity transformation also satisfies the hypotheses (namely the identity is an isometry and it fixes all points of H) and the identity is not a reflection. Indeed, the fixed point set of the identity transformation is \mathbb{E}^n whereas the fixed point set of a reflection is exactly a hyperplane (by the argument at the start of 1(b) but applied to general dimension).

Grading Scheme:

- 1 mark for (1).
- 2 marks for (2). There are various ways of arguing this. If your argument is completely rigorous you receive full points (even if it differs from the above), otherwise, depending on how flawed your logic is, you receive between 0 and 1.5 points.
- 1 mark for (3) if your logic is perfect, otherwise you lose at least 0.5 points.

QUESTION 2. (2 points) For each of the following statements, determine if the statement is true or false.

Circle **T** for the true statements and **F** for the false statements. For each correct answer you will receive 1 point. For each incorrect answer you will receive -1 points. If you do not circle any of the letters next to a statement, you will receive 0 points for that statement.

T F For every reflection $R_\ell : \mathbb{E}^2 \rightarrow \mathbb{E}^2$ by a line ℓ and every translation $T_{\vec{a}} : \mathbb{E}^2 \rightarrow \mathbb{E}^2$ with displacement vector \vec{a} we have $R_\ell \circ T_{\vec{a}} = T_{\vec{a}} \circ R_\ell$.

T F If $f : \mathbb{E}^n \rightarrow \mathbb{E}^n$ is an isometry, then f is one-to-one (that is, for every $p_1, p_2 \in \mathbb{E}^n$ if $f(p_1) = f(p_2)$ then $p_1 = p_2$).

The answers are given in order:

- False. For example if ℓ is the x -axis and $\vec{a} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, then $R_\ell \circ T_{\vec{a}} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ whereas $T_{\vec{a}} \circ R_\ell \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
- True. For every $p_1, p_2 \in \mathbb{E}^n$, we have $f(p_1) = f(p_2) \Rightarrow d(p_1, p_2) = d(f(p_1), f(p_2)) = 0 \Rightarrow p_1 = p_2$.

Grading Scheme: From top to bottom the answers are F,T. There is 1 mark for each correct answer, -1 mark for each incorrect answer, 0 marks if there is no answer. This produces a running total. If this total is positive, that's your score on the question. If this total is non-positive, your score is zero.

QUESTION 3. (2 points) Consider the line ℓ in the complex plane with equation

$$z = (2 + \mathbf{i}) + t(4 - \mathbf{i}), \quad t \in \mathbb{R}.$$

Find the intersection of ℓ with the y -axis. You should give the point of intersection as a complex number.

Solution: The intersection of a line with the y -axis is the point on the line which is imaginary, i.e., its real part vanishes. Now since $t \in \mathbb{R}$, the points on the line are of the form

$$(2 + 4t) + \mathbf{i}(1 - t).$$

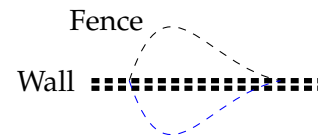
and the real part vanishes precisely when $t = -\frac{1}{2}$. Therefore the point of intersection is the complex number

$$z = (1 - t)\mathbf{i} = \left(1 + \frac{1}{2}\right)\mathbf{i} = \frac{3}{2}\mathbf{i}.$$

Grading Scheme:

- 1 pt for showing you understand the point of intersection is imaginary.
- 1 pt for calculating the point correctly and *giving it as a complex number*.

QUESTION 4. [3 points] You want to build a fence with its two ends attached to a wall such that the area enclosed between the wall and the fence is 10 square meters. What is the minimum possible length of the fence, and its shape when the length is minimized? You should justify your answer.



Let L be the length of the fence. Consider the reflection of the fence by the wall. The fence and its reflection form a closed non-intersecting curve of length $2L$ enclosing an area of 20 m^2 . By the isoperimetric inequality, the area A enclosed by the curve satisfies

$$4\pi A \leq (2L)^2 \Rightarrow 4\pi(20) \leq 4L^2 \Rightarrow \sqrt{20\pi} \leq L$$

Therefore the minimum length of the fence is $\sqrt{20\pi} = 2\sqrt{5\pi}$. The isoperimetric inequality also states that equality occurs exactly when the curve is a circle. Thus, the shape of the length-minimizing fence is semicircular.

Grading Scheme: 1 pt for reflecting the fence by the wall,
1 pt for using the isoperimetric inequality to compute the minimum of L ,
1 pt for mentioning that the optimal solution is achieved for the case of a semicircle.

QUESTION 5. (3 points) Let A, B, C, D be four points in the complex plane corresponding to the complex numbers $a, b, c, d \in \mathbb{C}$. (i) Show that AB and CD are parallel if and only if $(a - b)/(c - d)$ is real. (ii) How can you use the ratio $(a - b)/(c - d)$ to determine instead whether AB and CD are orthogonal? State your claim as an “if and only if” statement and then prove it.

Solution: *Aside: Assume “four points” refers to four distinct points, or at least assume $A \neq B$ and $C \neq D$, so that the lines mentioned in the question make sense.*

(i) AB and CD are said to be parallel iff their directing vectors $\vec{a} - \vec{b}$ and $\vec{c} - \vec{d}$ are parallel, i.e., $\vec{a} - \vec{b} = \lambda(\vec{c} - \vec{d})$, $\lambda \in \mathbb{R}$. As complex numbers this corresponds to $a - b = \lambda(c - d)$, $\lambda \in \mathbb{R}$. This is equivalent to $(a - b)/(c - d) \in \mathbb{R}$ because we know $c \neq d$.

(ii) We claim that AB and CD are orthogonal iff $(a - b)/(c - d)$ is purely imaginary. To prove this: note orthogonality occurs iff the directing vectors $\vec{a} - \vec{b}$ and $\vec{c} - \vec{d}$ are orthogonal. In other words, up to rescaling by $\lambda \in \mathbb{R}$, one vector should be a rotation of the other by $\pm\pi/2$. This condition can be stated in terms of complex numbers as $a - b = i\lambda(c - d)$, $\lambda \in \mathbb{R}$, since $i = e^{i\pi/2}$. Equivalently, $(a - b)/(c - d) \in i\mathbb{R}$ because we know $c \neq d$.

Grading Scheme:

- 1 pt for proving (i) rigorously
- 1 pt for stating the correct claim in (ii)
- 1 pt for proving the claim in (ii)