University of Ottawa
Dept. of mathematics and Statistics
Calculus III for engineers
MAT 2322 3X
Practice Exam for the Final
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They will be approximately 17 questions among which 10 are multiple choice questions. Don't forget to review the midterms and the practice tests.

I will update the solution if I find a problem.

1. Find the volume of the solid under the plane $x + 2y - z = 0$ and above the region D in the $xy-$plane bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.
Answer: 32/15. For more details see Example 1 page 1014.

2. Find the total arclength of the curve given by

$$\vec{r}(t) = (\cos(t), \sin(t), \ln(\cos(t))) \quad , \qquad 0 \leq t \leq \pi/4.$$

Answer: $\ln\left(1 + \sqrt{2}\right)$.

3. State the F.T.L.I. and use it to evaluate $\int_C \vec{F} \cdot \vec{r}$, where $\vec{F} = x\vec{\imath} + y\vec{\jmath}$ and C is the arc of of the parabola $y = 2x^2$ from $(1, 2)$ to $(3, 18)$. vfill

Answer: See the theorem in section 16.3.
A potential function for $\vec{F}$ is $f(x, y) = \frac{x^2+y^2}{2}$. So, by the F.T.L.I. we have $\int_C \vec{F} \cdot \vec{r} = f(3, 18) - f(1, 2)$.

4. Consider the vector field $\vec{F}(x, y, z) = (xy^2 + xz^2)\,\vec{\imath} + (x^2y + yz^2)\,\vec{\jmath} + (y^2z + x^2z)\vec{k}$.

(a) Show that $\vec{F}$ is conservative.

(b) (Bonus) Find a potential function for $\vec{F}$, i.e. find $f(x, y, z)$ such that $\vec{F}(x, y, z) = \vec{\nabla}\, f(x, y, z)$

Answer: (a) rot$\vec{F} = \vec{0} \Rightarrow \vec{F}$ conservative.
(b) $f(x, y, z) = (x^2z^2 + x^2y^2 + y^2z^2)/2 + K$.

5. Find and classify the critical points of the function $f(x, y) = (y^2 + x^2)e^{y^2 - x^2}$.

Answer: $(0, 0)$ local min, $(\pm 1, 0)$ saddle pts.

6. Compute the following double integral $\int_0^1 \int_x^1 \cos(\pi y^2)\, dy\, dx$. Hint: inverse the order of integration.

Answer: 0.

7. Let $\vec{F}(x,y) = (e^{x^2} - \cos(y))\,\vec{\imath} + (y^3 + x\sin(y) + 2xy)\,\vec{\jmath}$ and let C be the closed curve composed of the half-circle $x^2 + y^2 = 1$, $y \geq 0$, and the straight segment from the point (-1, 0) to the point (1,0) oriented counterclockwise. Compute the line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

Answer: Use Green's theorem and polar coords to find 4/3.

8. Let E be the 3-dimensional solid **in the first octant** bounded from outside by the sphere $x^2 + y^2 + z^2 = 1$ and from inside by the cone $z = \sqrt{x^2 + y^2}$. If the mass density of this solid is $\delta(x, y, z) = x + y$, find the total mass of this solid?
Hint: use spherical coords.

Answer: $\frac{\pi - 2}{16}$. (Check this...)

9. Let S be the positively (outward) oriented surface composed of the part of the cone $z = \sqrt{x^2+y^2}$ for $1 \le z \le 2$ and consider the vector field

$$\vec{F} = 2xz\vec{i} - yz\vec{j} + (z-3y^2)\vec{k}\,.$$

Compute $\iint_S \vec{F}d\vec{S}$.

Answer: $31\pi/3$.

10. Consider the parametric surface S given by

$$\vec{r}(\theta,\phi) = \cos(\theta)\sin(\phi)\,\vec{i} + \sin(\theta)\sin(\phi)\,\vec{j} + \cos(\phi)\,\vec{k}, \quad 0 \le \theta \le \frac{\pi}{4},\, 0 \le \phi \le \frac{\pi}{2}\,.$$

Let $f(x,y,z) = x^2+y^2$. What is the value of the surface integral $\displaystyle\iint_S f\,dS$?

Answer: $\pi/6$.

11. Let E be the solid in the first octant bounded by $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$. If the mass density $\delta(x, y, z) = x + y$, find the total mass of this solid.

Answer: Use cylindrical coords to find 256/15.

12. Find the absolute extrema of the function $f(x,y) = 2x^2 + y^2$ on the disk centred at the point (0,1) and radius 2.

Answer: Abs Min: (0, 0). Abs Maxs: $(-\sqrt{3}, 2)$, $(\sqrt{3}, 2)$ (you also get these points (0,-1), (0,3) if using Lagrange multiplier).

13. Consider the vector field $\vec{F}(x, y, z) = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$.
(a) Compute the divergence of $\vec{F}$, i.e. compute div$\vec{F} = \vec{\nabla} \cdot \vec{F}$.
(b) **Use the divergence thm** to compute the surface (flux) integral $\int\int_S \vec{F} \cdot d\vec{S}$, where S is sphere centred at the origin and radius a oriented positively. (c) Compute directly $\int\int_S \vec{F} \cdot d\vec{S}$ for S and $\vec{F}$ as above.

Answer: (a) $div\vec{F} = 3$.
(b),(c) $4\pi a^3$.

14. Sketch the region of integration and then compute the following double integral

$$\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y)\,dx\,dy$$

Hint: Use another coords system (polar).

Answer: $2\sqrt{2}/3$.

15. Consider the vector field $\vec{F} = 2xe^{x^2}\sin(y)\vec{\imath} + e^{x^2}\cos(y)\vec{\jmath}$.

(a) Show that $\vec{F}$ is conservative.

(b) Find a potential function f; i.e. find f such that $\nabla f = \vec{F}$.

(c) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the arc of the circle parametrized by $\vec{r} = \cos(\theta)\vec{\imath} + \sin(\theta)\vec{\jmath}, \quad -\pi/2 \le \theta \le \pi/2$.

Answer: (a) $P_y = Q_x$.
(b) $f(x, y) = e^{x^2}\sin(y)$.
(c) Use the F.T.L.I. to find $\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(\pi/2)) - f(\vec{r}(-\pi/2)) = \textit{find the numerical value}$.

16. Consider the vector field

$$\vec{F}(x,y,z) = (yz^3 - 2y)\,\vec{i} + (xz^3 + 2x)\,\vec{j} + (3xyz^2 + z^4)\,\vec{k}$$

and let C the circle $x^2 + y^2 = 9$ on the plane $z = 0$ oriented positively when viewed from above. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$.

Answer: Stoke's thm in this case gives a simpler computation 36π.