PHY2323 Electricity and Magnetism Assignment 1 Solution

Q1:

Gauss's law in terms of electric field expressed as:

$$\oint \vec{E} \cdot \vec{ds} = \frac{Q_{enc}}{\epsilon_0} \tag{1}$$

Electrical field has only radial component due to symmetry:

$$\vec{E} = E_r \hat{r} \tag{2}$$

$$\oint \vec{E} \cdot \vec{ds} = E_r(4\pi r^2) = \frac{Q_{enc}}{\epsilon_0} \tag{3}$$

$$E_r = \frac{Q_{enc}}{4\pi r^2 \epsilon_0} \tag{4}$$

enclosed charge Q_{enc} by the sphere with radius r is:

$$Q_{enc} = \int_v \rho(r')dv' = \int_0^r \int_0^\pi \int_0^{2\pi} (\frac{q}{4\pi a^2 r'} e^{-\frac{r'}{a}}) r'^2 \sin\theta' dr' d\theta' d\phi' \tag{5}$$

$$Q_{enc} = \frac{q}{a} \int_0^r \frac{r'}{a} e^{-\frac{r'}{a}} dr' \tag{6}$$

we change the integration variable:

$$u = \frac{r'}{a} \quad , \quad du = \frac{1}{a} dr' \tag{7}$$

$$Q_{enc} = q \int u e^{-u} du \tag{8}$$

by using the method of integration by parts we have:

$$Q_{enc} = q(-ue^{-u} - \int -e^{-u} du) = q(-ue^{-u} - e^{-u}) = q(-\frac{r'}{a} e^{-\frac{r'}{a}} - e^{-\frac{r'}{a}}) \Big|_0^r = q(-\frac{r}{a} e^{-\frac{r}{a}} - e^{-\frac{r}{a}} + 1) \tag{9}$$

so when $r \to \infty$ enclosed charge Q_{enc} becomes q and when it $r \to 0$, Q_{enc} is equal to zero. Substituting above result for Q_{enc} in equation (4) we can determine the equation for radial electric field:

$$\boxed{E_r = \frac{Q_{enc}}{4\pi r^2 \epsilon_0} = \frac{q}{4\pi r^2 \epsilon_0}(-\frac{r}{a}e^{-\frac{r}{a}} - e^{-\frac{r}{a}} + 1)} \tag{10}$$

Q2: Electric field due to the charge distribution:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho_s\ (\vec{r} - \vec{r}')ds'}{|\vec{r} - \vec{r}'|^3} \tag{11}$$

$$r = 0 \quad , \quad r' = 2\hat{r} \tag{12}$$

$$ds' = {r'}^2 \sin\theta' d\theta' d\phi' = 4 \sin\theta' d\theta' d\phi' \tag{13}$$

$$\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \frac{-2}{8}\hat{r} \tag{14}$$

where r represent the position that we want to measure the electric field and r' is the points in the surface distribution. So electric field is:

$$\vec{E} = -\frac{1}{4\pi\epsilon_0} \int\int \hat{r}\sigma_0 \cos\theta' \sin\theta' d\theta' d\phi' \tag{15}$$

We must express $\hat{r}$ with x-y-z components. Transform from Cartesian to Spherical Coordinate can be done using the following relation:

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{16}$$

so $\hat{r}$ is:

$$\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \tag{17}$$

knowing that only the z component is nonzero we can write:

$$\vec{E} = -\frac{\sigma_0}{4\pi\epsilon_0} \int_0^{\pi} \cos^2\theta' \sin\theta' d\theta' \int_0^{2\pi} d\phi' \hat{z} = -\frac{\sigma_0}{2\epsilon_0} \int_0^{\pi} \cos^2\theta' \sin\theta' d\theta' \hat{z} \tag{18}$$

by change of variables we can integrate:

$$u = \cos\theta' \quad , \quad du = -\sin\theta' d\theta' \tag{19}$$

$$\vec{E} = -\frac{\sigma_0}{2\epsilon_0}\int_0^\pi \cos^2\theta' \sin\theta' \hat{z} = \frac{\sigma_0}{2\epsilon_0}\int u^2\, du\, \hat{z} \tag{20}$$

$$= \frac{\sigma_0}{2\epsilon_0}\frac{u^3}{3} = \frac{\sigma_0}{2\epsilon_0}\frac{\cos\theta'^3\ \hat{z}}{3}\Bigg|_0^\pi \tag{21}$$

$$\boxed{\vec{E} = -\frac{\sigma_0}{3\epsilon_0}\hat{z}} \tag{22}$$

Q3:

The electric flux passing through the surface s:

$$\psi = \epsilon \int \vec{E}\cdot\vec{ds} \tag{23}$$

$$\vec{ds} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{r} = 4\sin\theta\, d\theta\, d\phi\, \hat{r} \tag{24}$$

$$\psi = \epsilon \int \vec{E}\cdot\vec{ds} = \epsilon \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}} (25\hat{x})\cdot(4\sin\theta\, d\theta\, d\phi\, \hat{r}) = 100\,\epsilon \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}} (\hat{x}\cdot\hat{r})\sin\theta\, d\theta\, d\phi \tag{25}$$

Using equation (17) we have:

$$\psi = 100\,\epsilon \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}} \hat{x}\cdot(\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z})\sin\theta\, d\theta\, d\phi$$

$$= 100\,\epsilon \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}} \sin^2\theta\cos\phi\, d\theta\, d\phi = 100\,\epsilon \int_0^{\frac{\pi}{4}} \sin^2\theta\, d\theta = 100\,\epsilon \int_0^{\frac{\pi}{4}} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= 100\,\epsilon\,(\frac{\theta}{2} - \frac{\sin 2\theta}{4})\Bigg|_0^{\frac{\pi}{4}} = 100\,\epsilon\,(\frac{\pi}{8} - \frac{1}{4})$$

$$\boxed{\psi = 100\,\epsilon\,(\frac{\pi}{8} - \frac{1}{4})} \tag{26}$$

Q4:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_{s'} \frac{\rho_s(\vec{r}-\vec{r}')\,ds'}{|\vec{r}-\vec{r}'|^3} \tag{27}$$

$$ds' = \rho'\,d\rho'\,d\phi' \tag{28}$$

We know that the density is $\rho_S = k$ for $a < \rho < b$ and it is constant. So $\vec{r}$ and $\vec{r'}$ is going to be:

$$\vec{r} = z\hat{z} \quad , \quad \vec{r}' = \rho'\hat{\rho}$$

and consequently we have:

$$|\vec{r}-\vec{r}'|^3 = (z^2+\rho'^2)^{\frac{3}{2}} \quad , \quad \vec{r}-\vec{r}' = z\hat{z} - \rho'\hat{\rho}$$

Thus for equation (27) we obtain:

$$\vec{E} = \frac{k}{4\pi\epsilon_0}\int_a^b\int_{\frac{\pi}{2}}^{\pi} \frac{(z\hat{z}-\rho'\hat{\rho})\,d\rho'\,d\phi'}{(z^2+\rho'^2)^{\frac{3}{2}}} \tag{29}$$

solving only for the z component:

$$E_z = \frac{k}{4\pi\epsilon_0}\int_a^b\int_{\frac{\pi}{2}}^{\pi} \frac{z\,d\rho'\,d\phi'}{(z^2+\rho'^2)^{\frac{3}{2}}} = \frac{k}{4\pi\epsilon_0}(\pi-\frac{\pi}{2})\int_a^b \frac{z\,d\rho'}{(z^2+\rho'^2)^{\frac{3}{2}}} \tag{30}$$

$$= -\frac{kz}{8\epsilon_0}\frac{1}{(z+\rho'^2)^{\frac{3}{2}}}\Bigg|_a^b = \frac{-kz}{8\epsilon_0}(\frac{1}{(z^2+b^2)^{\frac{1}{2}}} - \frac{1}{(z^2+a^2)^{\frac{1}{2}}}) \tag{31}$$

$$\boxed{E_z = \frac{kz}{8\epsilon_0}(\frac{1}{\sqrt{z^2+a^2}} - \frac{1}{\sqrt{z^2+b^2}})} \tag{32}$$